Summer term 2022

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Symplectic Geometry

Sheet 2

Exercise 1. Let V be a real vector space of dimension $2n$, $J \in GL(V)$ a complex structure on V, and $W \subset V$ a totally real subspace of dimension n. Let $\Omega(V, J, W) \subset \bigwedge^2 V^*$ be the space of symplectic forms which are compatible with J and for which W is Lagrangian. Prove that $\Omega(V, J, W)$ can be identified with the space of inner products on W, and conclude that $\Omega(V, J, W)$ is contractible.

Exercise 2. Let (E, ω) be a symplectic vector bundle of rank $2n$ over a manifold M. Let ω_{std} be the standard symplectic structure on $\mathbb{R}^{2n}$. Show that for every $x \in M$ there is an open neighbourhood $U \subset M$ of x and a trivialisation $\varphi : E|_U \cong U \times \mathbb{R}^{2n}$ satisfying $\varphi^*(\omega_{\text{std}}) = \omega$. *Hint: Construct sections which form a symplectic basis in each fibre.*

Exercise 3. Find a symplectic vector bundle (E, ω) with a non-orientable Lagrangian subbundle $L \subset E$. *Hint: It is enough to consider vector bundles over S^1.*

Exercise 4. Show that there are symplectic vector bundles (E, ω) of real rank 4 which do not admit Lagrangian subbundles.

Please hand in your solutions in the lecture on Friday, 13 May 2022.