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Functional Analysis II
Assignment 10

Problem 37. Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove:

(*i*) The operator $|A| = \sqrt{A^*A}$ constructed with Hilbert space techniques (see Problem 18) coincides with $|A|$ defined via the functional calculus.

(*ii*) $A \leqslant |A|$.

(*iii*) There exists a unique pair $A_+, A_- \in \mathcal{B}(\mathcal{H})$ of self-adjoint operators such that

$$A_+, A_- \geqslant \mathbb{O}\ , \quad A_+A_- = \mathbb{O}\ , \quad A = A_+ - A_-\ .$$

Problem 38 (Operator monotone functions). A continuous real-valued function f on an interval I is said to be *operator monotone* (on the interval I), if $A \leqslant B$ implies that $f(A) \leqslant f(B)$ for all self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $\sigma(A) \subset I$ and $\sigma(B) \subset I$.

(*i*) Prove that f_α given by $f_\alpha(t) := \frac{t}{1+\alpha t}$ is operator monotone on $\mathbb{R}_+$ if $\alpha \geqslant 0$.

(*ii*) Let $\alpha \in [0,1]$, and $A, B \in \mathcal{B}(\mathcal{H})$ be such that $0 \leqslant A \leqslant B$. Prove that $0 \leqslant A^\alpha \leqslant B^\alpha$, i.e. that $x \mapsto x^\alpha$ is operator monotone on $\mathbb{R}_+$. [*Hint:* You may want to use that for all $x \geqslant 0$ we have $x^\alpha = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{x}{x+\lambda} \frac{d\lambda}{\lambda^{1-\alpha}}$ for all $\alpha \in (0,1)$.]

(*iii*) Find a counterexample for (*ii*) when $\alpha > 1$.

Problem 39. Let $A : L^2([0,1]) \to L^2([0,1])$ be given by $Af(x) := xf(x)$ for a.e. $x \in [0,1]$.

(*i*) Prove that $A = A^*$, $\|A\| = 1$ and $\sigma(A) = [0,1]$.

(*ii*) Give the explicit action of $f(A) \in \mathcal{B}(L^2([0,1]))$ for any bounded measurable function $f : [0,1] \to \mathbb{C}$.

(*iii*) For any $\psi \in L^2([0,1])$ express $(\psi, f(A)\psi)$ as an integral with respect to the measure $\Omega \mapsto (\psi, E_\Omega \psi)$, where E denotes the projection-valued measure given by A.

Problem 40. Let $\mathcal{H}$ be a Hilbert space, let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and let E denote the projection-valued measure given by A. Prove:

(*i*) For any Borel set $\Omega \subset \sigma(A)$, the subspace $R(E_\Omega)$ is invariant under A .

(*ii*) If $\Omega \subset \sigma(A)$ is closed, then $\sigma(A|_{R(E_\Omega)}) \subset \Omega$.