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Functional Analysis II
Assignment 8

Problem 29. Let $\mathcal{H}$ be a Hilbert space and $A = A^* \in \mathcal{B}(\mathcal{H})$. Prove:

(*i*) $A \leqslant \|A\| \, \mathbb{I}$.

(*ii*) If $A \geqslant \mathbb{O}$ then $\sigma(A) \subset [0, \|A\|]$.

(*iii*) If $\sigma(A) \subset [0, R]$ for some $R > 0$, then $\mathbb{O} \leqslant A \leqslant R\,\mathbb{I}$.

Problem 30. Let $\mathcal{H}$ be a Hilbert space, let $S, T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and assume that $TS = ST$. Show for any bounded Borel function $f \in \mathcal{M}_b(\sigma(T))$ that $f(T)S = Sf(T)$.

Problem 31. Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove:

(*i*) If $A \leqslant B$ then $C^*AC \leqslant C^*BC$ for all $C \in \mathcal{B}(\mathcal{H})$.

(*ii*) If $\mathbb{O} \leqslant A \leqslant B$ then $\|A\| \leqslant \|B\|$.

(*iii*) If $A \geqslant \mathbb{O}$, then A is invertible iff $A \geqslant c\,\mathbb{I}$ for some $c > 0$.

Problem 32. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be normal. Prove:

(*i*) $N(T) = N(T^*)$.

(*ii*) $\overline{R(T)} = \overline{R(T^*)}$, and if $R(T)$ is closed then $R(T^*) = R(T)$.

(*iii*) If T has a bounded one-sided inverse then T is invertible.