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Functional Analysis II
Assignment 7

Problem 25. Let $\mathcal{H}$ be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. Prove:

(*i*) There exist unique self-adjoint operators $R_A, I_A \in \mathcal{B}(\mathcal{H})$ such that $A = R_A + iI_A$.

(*ii*) A is normal iff $[R_A, I_A] := R_A I_A - I_A R_A = \mathbb{O}$.

(*iii*) A is unitary iff A is normal and $R_A^2 + I_A^2 = \mathbb{I}$.

(*iv*) If $T = T^*$ and $\|T\| \leqslant 1$, then $U := T + i\sqrt{\mathbb{I} - T^2}$ is unitary and $T = \frac{1}{2}(U + U^*)$.

(*v*) There exist unitary operators $U_1, \dots, U_4$ and $a_1, \dots, a_4 \in \mathbb{C}$ such that

$$A = a_1 U_1 + a_2 U_2 + a_3 U_3 + a_4 U_4 \,,$$

and $|a_j| \leqslant \|A\|/2$ for all j.

Problem 26 (Weyl sequences – II). Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Prove:

(*i*) If $\lambda \in \sigma(T)$ then there exists a Weyl sequence for T at λ *or* for T^* at $\overline{\lambda}$.

(*ii*) If T is normal ($T^*T = TT^*$), then $\lambda \in \sigma(T)$ iff T has a Weyl sequence at λ.

(*iii*) If T is self-adjoint and λ is an isolated point in $\sigma(T)$ then λ is an eigenvalue of T.

Problem 27. Let A be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. For $n \in \mathbb{Z}$ let λ_n denote its eigenvalues labeled such that they may be repeated due to multiplicity and

$$\lambda_{-1} \leqslant \lambda_{-2} \leqslant \cdots < 0 < \cdots \leqslant \lambda_2 \leqslant \lambda_1 \,.$$

Prove that for each $n \in \mathbb{N}$

$$\lambda_n = \inf_{\mathcal{H}_{n-1}} \sup_{\substack{x \perp \mathcal{H}_{n-1} \\ \|x\|=1}} \langle x, Ax \rangle \,, \quad \lambda_{-n} = \sup_{\mathcal{H}_{n-1}} \inf_{\substack{x \perp \mathcal{H}_{n-1} \\ \|x\|=1}} \langle x, Ax \rangle \,,$$

where $\inf_{\mathcal{H}_{n-1}}$ and $\sup_{\mathcal{H}_{n-1}}$ are over all possible $(n-1)$-dimensional subspaces $\mathcal{H}_{n-1}$ of $\mathcal{H}$.

Problem 28 (Volterra integral operator – II). Let $V : L^2[0,1] \to L^2[0,1]$ be the Volterra integral operator introduced in Problem 20, i.e. $Vf(x) = \int_0^x f(y)\,dy$.

(i) Show that if $f \in L^2[0,1]$ is an eigenfunction of the operator V^*V with eigenvalue λ, then $\lambda > 0$, f is twice differentiable a.e., and $\lambda f'' + f = 0$ a.e. in $[0,1]$.

[*Hint:* You may use without proof that if f is integrable then $x \mapsto \int_0^x f(y)dy$ is a.e. differentiable with derivative f (Lebesgue differentiation theorem).]

(ii) Find the collection $\{\lambda_n\}_{n=1}^\infty$ of all eigenvalues of V^*V, and check that the corresponding family of eigenfunctions $\{f_n\}_{n=1}^\infty$ is (up to normalization) an ONB in $L^2[0,1]$.

[*Note:* This exercise is intended to be done without using the spectral theorem for normal compact operators.]

(iii) Deduce from (ii) that $\|V\| = \frac{2}{\pi}$.