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Functional Analysis II

Assignment 4

Problem 13 (Spectrum of the product). Let X be a Banach space and $S, T \in \mathcal{B}(X)$.

(*i*) Prove that $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$.

(*ii*) Show that $\sigma(TS) = \sigma(ST)$ is not true in general.

Problem 14 (Spectrum of self-adjoint operators). Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, i.e. $A^* = A$. Prove the following:

(*i*) $\sigma(A) \subset \Big[\inf\limits_{x\in\mathcal{H}, \|x\|=1} \langle x, Ax\rangle \, , \sup\limits_{x\in\mathcal{H}, \|x\|=1} \langle x, Ax\rangle \Big] \subset \mathbb{R}$.

(*ii*) $\sigma_r(A) = \emptyset$.

(*iii*) If $x, y \in \mathcal{H}$ and $\lambda \neq \mu$ are such that $Ax = \lambda x$ and $Ay = \mu y$ then $\langle x, y\rangle = 0$.

(*iv*) If $\sigma(A) = \{0\}$ then $A = \mathbb{O}$.

Problem 15 (Weyl sequences). Let X be a Banach space and $T \in \mathcal{B}(X)$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called a *Weyl sequence* of T at $\lambda \in \mathbb{C}$, if $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|Tx_n - \lambda x_n\| \to 0$ as $n \to \infty$. Prove:

(*i*) If T has a Weyl sequence at $\lambda \in \mathbb{C}$ then $\lambda \in \sigma(T)$.

(*ii*) If $\lambda \in \partial\sigma(T)$ then T has a Weyl sequence at $\lambda \in \mathbb{C}$.

Now, let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint.

(*iii*) Prove that T has a Weyl sequence at λ iff $\lambda \in \sigma(T)$.

Problem 16 (Multiplication operators II). Let (X, μ) be a σ-finite measure space, let $1 \leqslant p < \infty$, and for a measurable function $h : X \to \mathbb{C}$ let

$$\Omega_h := \{f \in L^p(X, \mu) : hf \in L^p(X, \mu)\}\,.$$

Let $M_h : \Omega_h \to L^p(X, \mu)$, $f \mapsto hf$.

(*i*) Prove that $M_h \in \mathcal{B}(L^p(X, \mu))$ iff $h \in L^\infty(X, \mu)$.

Assuming $h \in L^\infty(X, \mu)$ prove the following:

(*ii*) $\sigma_p(M_h) = \big\{\lambda \in \mathbb{C} : \mu(\{x \in X : h(x) = \lambda\}) > 0\big\}$.

(*iii*) $\rho(M_h) = \big\{\lambda \in \mathbb{C} : \exists c > 0 \text{ such that } |\lambda - h(x)| \geqslant c \text{ a.e.}\big\}$.

For more details please visit http://www.math.lmu.de/~gottwald/15FA2/