LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis II
Assignment 3

Problem 9 (Projections II).

(i) Show that for every linear subspace $W \subset V$ of a linear space V there exists a projection $P : V \to V$ with $R(P) = W$. [*Hint:* Zorn.]

(ii) Find a normed space X and a projection $P : X \to X$ that is not continuous.

Now let X be a Banach space and assume that $X = X_0 \oplus X_1$ for some closed subspaces X_0 and X_1 of X. Prove the following:

(iii) There exists a bounded projection $P : X \to X$ with $N(P) = X_0$ and $R(P) = X_1$.

(iv) If $\dim X_1 < \infty$ and $\tilde{X}_1$ is another subspace of X such that $X_0 \cap \tilde{X}_1 = \{0\}$ then $\dim \tilde{X}_1 \leqslant \dim X_1$, and if $X_0 \oplus \tilde{X}_1 = X$ then $\dim \tilde{X}_1 = \dim X_1$, i.e. the codimension of a closed subspace of X is well-defined.

Problem 10.

(i) Show that a compact operator $T : X \to X$ on an infinite-dimensional Banach space X is never surjective (compare with P2 (iv) and P4 (ii)).

(ii) Let $1 \leqslant p < \infty$ and $T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $x \mapsto (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$. Find $y \in \ell^p(\mathbb{N})$ such that $Tx = y$ has no solution $x \in \ell^p(\mathbb{N})$. Why does such a y exist?

Problem 11. Let X and Y be Banach spaces and let $T : X \to Y$ be bounded.

(i) Assume that $\|Tx\|_Y \geqslant c\|x\|_X$ for all $x \in X$ and some $c > 0$ and show that in this case T can be compact only if $\dim X < \infty$.

Now let $T : X \to Y$ be compact and $\dim X = \infty$.

(ii) Prove that $0 \in \overline{T(S)}$, where $S := \{x \in X : \|x\| = 1\}$.

(iii) Construct a non-injective compact operator arbitrarily close in norm to T.

Problem 12 (Shift operator). Let $T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$, $x \mapsto (x_2, x_3, x_4, \dots)$.

(i) Prove that $T \in \mathcal{B}(\ell^1(\mathbb{N}))$ and determine $\|T\|$.

(ii) Find the adjoint T' (domain and action).

(iii) Determine the spectra, point spectra and the residual spectra of T and T'.