Functional Analysis

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August 29, 2015

Lecture:
Title Functional Analysis
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Term summer term 2015
This document:
Version of 2015-08-29
Based on notes of a student during lecture

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Contents

1 Historical Perspective

2015-04-14

Roots:

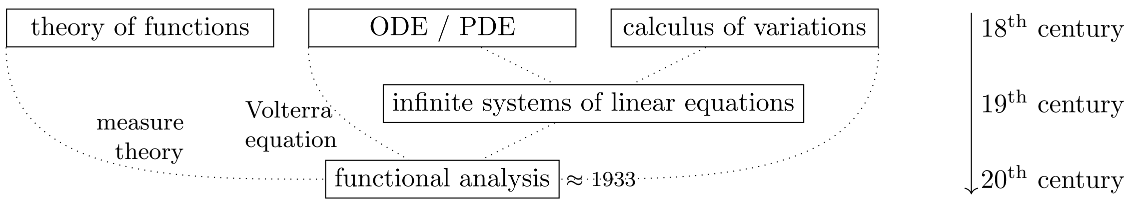


Figure 1: History of functional analysis

What is functional analysis?

- Study of functional dependencies between (topological) spaces
- Study of spaces of functions
- Language of PDF / calculus of cariations, numerical analysis
- Language of quantum mechanics

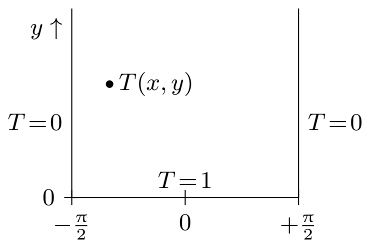
Shift in mathematics between $19^{\text{th}}/20^{\text{th}}$ century:

- Volterra's speech on 1900 International Congress of Mathematicians in Paris:
  "19^{th} cenutry math is about the study of a *single* function."
  I.e. definition of a function, continuity, differentiability
- Typical 19^{th} century math:
  Theorem 1.1 *(Weierstrass 1872)*. A function $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),\ 0 < a < 1,\ b \in \{2n+1 \mid n \in \mathbb{N}\}$ is continuous but nowhere differentiable. □
- Special functions:
  - Bessel function: $J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot \Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}$
  - Hermite polynomial: $H_n(x) = (-1)^n \mathrm{e}^{+x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}$
- Functional analysis shifted the view to the study of sets of functions:

  $$\text{definition of continuity} \quad \longrightarrow \quad \text{properties of sets of continuous functions}$$

  First theorem: Arzela-Ascoli theorem (coming soon).

Example 1.2 *(temperature distribution on an infinite slab)*.



If $T(x, y)$ is the temperature at a point (x, y), then:

(1) $\frac{\partial^2}{\partial^2 x} T(x, y) + \frac{\partial^2}{\partial^2 y} T(x, y) = 0$

(2) $\forall y \in]0, \infty[:\ T(-\frac{\pi}{2}, y) = 0 = T(+\frac{\pi}{2}, y)$

(3) $\forall x \in]-\frac{\pi}{2}, +\frac{\pi}{2}[:\ T(x, 0) = 1$

We guess

$$T : \left]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}\right[\to]0, \infty[, \quad T(x, y) = \sum_{n=0}^{\infty} x_n \mathrm{e}^{-(2n+1)y} \cos((2n+1)x),$$

this automatically satisfies (0) and (1). For (b) we get the equation

$$1 = \sum_{n=0}^{\infty} x_n \cos((2n+1)x), \quad x \in \left]-\tfrac{\pi}{2}, +\tfrac{\pi}{2}\right[.$$

By subsequent differentiating and putting $x = 0$ we get:

$$\begin{aligned} 1 &= x_0 + 3^0 x_1 + 7^0 x_2 + \dots \\ 0 &= x_0 + 3^2 x_1 + 7^2 x_2 + \dots \\ 0 &= x_0 + 3^4 x_1 + 7^4 x_2 + \dots \end{aligned}$$

◊

We got a set of equations of the form:

$$\sum_{n=1}^{\infty} a_{nm} x_m = y_n \quad \text{i.e.} \quad \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} \tag{$*$}$$

Problem:

$$\sum_{n=1}^{\infty} a_{nm} x_m = y_n, \quad a_{nm}, y_n \in \mathbb{F}\ (= \mathbb{R} \text{ or } \mathbb{C}) \text{ given}, \quad x_m \text{ unknown}$$

How to solve it: 19^{th} century: finite approximations:

$$\text{pick } N: \qquad N\text{-th approximation } \sum_{n=1}^{N} a_{nm} x_m^{(N)} = y_n,\ n = 1, \dots, N \quad \Longrightarrow \quad \text{get } x_m^{(N)} \xrightarrow[n\to\infty]{\text{take}} x_m$$

Example 1.3.

Consider the following system:

$$\begin{aligned} x_1 + x_2 + \dots \quad &= 1 \\ x_2 + x_3 + \dots \quad &= 1 \\ x_3 + x_4 + \dots &= 1 \end{aligned}$$

Then:

$$\begin{aligned} \text{for odd } N\text{:} \quad x^{(N)} &= (1, 0, 1, 0, \dots) \\ \text{for even } N\text{:} \quad x^{(N)} &= (0, 1, 0, 1, \dots) \end{aligned}$$

By looking: $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$.

◊

Options one can encounter:

(A) $x^{(N)}$ does converge, and the limit is a solution of eq. ($*$)

(B) $x^{(N)}$ does not converge, but eq. ($*$) has a solution

(C) $x^{(N)}$ does not converge, and eq. ($*$) has no solution

(D) $x^{(N)}$ does converge, but eq. ($*$) has no solution

Question: What is the problem we are facing?

Recall that we studied equations

$$\sum_{m=1}^{\infty} a_{nm} x_m = y_n \quad \leftrightsquigarrow \quad Ax = y.$$

Here is one more example that leads to such an equation:

Example 1.4 *(Volterra equation)*. Let $K\colon [0,1] \times [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ be continuous functions. The *Volterra equation* is:

$$\begin{aligned} \text{Volterra equation of } 1^{\text{st}} \text{ kind:} \qquad & \int_0^s K(s,t) \cdot f(t)\, \mathrm{d}t = g(s) \\ \text{Volterra equation of } 2^{\text{nd}} \text{ kind:} \quad f(s) - & \int_0^s K(s,t) \cdot f(t)\, \mathrm{d}t = g(s) \end{aligned}$$

Riemann integration: Divide $[0,1]$ into N subintervals, $t_n^{(N)} = \frac{n}{N}, n = 0, \dots, N$:

$$\int_0^{t_n^{(N)}} K(t_n^{(N)}, t)\ f(t)\, \mathrm{d}t = \sum_{m=1}^{N} K(t_n^{(N)}, t_m^{(N)})\ f(t_m^{N})\ \tfrac{1}{N} + o(\tfrac{1}{N})$$

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Volterra equation of 1st kind:

$$\begin{array}{ccccccccc} a_{11}^{(N)}x_1^{(N)} & + & a_{12}^{(N)}x_2^{(N)} & + & \cdots & + & a_{1N}^{(N)}x_N^{(N)} & = & y_1^{(N)} \\ a_{21}^{(N)}x_1^{(N)} & + & a_{22}^{(N)}x_2^{(N)} & + & \cdots & + & a_{2N}^{(N)}x_N^{(N)} & = & y_2^{(N)} \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{N1}^{(N)}x_1^{(N)} & + & a_{N2}^{(N)}x_2^{(N)} & + & \cdots & + & a_{NN}^{(N)}x_N^{(N)} & = & y_N^{(N)} \end{array} \qquad \begin{array}{l} a_{nm}^{(N)} = K(t_n^{(N)}, t_m^{(N)})\frac{1}{N} \\ x_m^{(N)} = f(t_m^{(N)}) \\ y_n^{(N)} = g(t_n^{(N)}) \end{array}$$

Now:

$$\sum_{m=1}^{\infty} a_{nm}^{(N)} x_m^{(N)} = y_n^{(N)}, \qquad x_{\lfloor tN \rfloor}^{(N)} \overset{N\to\infty}{\longrightarrow} f(t),\ t \in \,]0,1[\qquad \Diamond$$

Historical perspective – overview:

(1)	$\sum_{m=1}^{\infty} a_{nm}x_m = y_n$	is linear $Ax = y$	where $x \leftrightsquigarrow (x_n)_{n=1}^{\infty}$
(2)	$\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$	is linear $Ax = y$	where $x \leftrightsquigarrow u(x,y)$
(3)	$\int_0^s K(s,t)\ f(t)\,\mathrm{d}t = g(s)$	is linear $Ax = y$	where $x \leftrightsquigarrow f(t)$

Problems:

(1) Notion of solution

(2) Continuity with respect to data

Concerning the continuity with respect to data:

Prop. 1.5. Let $A(t) = (a_{ij}(t))_{i,j=1}^{n}$ be a matrix that depends smoothly on t (smooth family), and vectors $y(t) = (y_j(t))_{j=1}^{n}$ smoothly on t. Suppose in addition $\forall t:\ \ker A(t) = \{0\}$. Then the solution $x(t)$ of $A(t)x(t) = y(t)$ depends smoothly on t. □

Proof. Observe $\det A$ is a smooth function:

$$\det A = \sum_{\pi}(-1)^{\operatorname{sgn}\pi} a_{1,\pi(1)}a_{2,\pi(2)}\dots a_{n,\pi(n)}, \quad x_j = \frac{\det\|\cdot\|}{\det A(t)} \quad \therefore \quad \det A \text{ is a smooth function} \qquad \blacksquare$$

Chapters:

- Normed linear spaces, Banach spaces, Hilbert spaces
- Linear operators on Banach spaces, dual spaces
- little bit more topology
- Three big results in functional analysis: Hahn-Banach theorem, Banach-Steinhaus theorem, open mapping principle
- Geometry of Banach space
- Compact operators and spectrum

Furthermore, let in the following be $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

2 Normed Linear Spaces

2015-04-17

2.1 Linear Spaces

Definition 2.1 *(linear space)***.** Set X equipped with two operations, on which two operations

$$\begin{aligned} \text{addition} \quad & X \times X \to X, \ (x,y) \mapsto x+y \\ \text{multiplication by scalar} \quad & \mathbb{F} \times X \to X, \ (\lambda, x) \mapsto \lambda \cdot x \end{aligned}$$

is called *linear space* over field $\mathbb{F}$, provided the following axioms are satisfied for any $x, y, z \in X$ and $a, b \in \mathbb{F}$:
Group structure:

- associativity: $(x+y)+z = x+(y+z)$
- identity element: $x + 0 = x$
- existence of inverses: $x + (-x) = 0$
- commutativity: $x + y = y + x$

Compatibility with field:

- compatibility of mul.: $a \cdot (b \cdot x) = (ab) \cdot x$
- compatibility of one: $1 \cdot x = x$
- distributivity I: $a \cdot (x+y) = a \cdot x + a \cdot y$
- distributivity II: $(a+b) \cdot x = a \cdot x + b \cdot x$

Example 2.2 *(examples of linear spaces)***.**

(1) Finite-dimensional euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$

(2) Space of inifinite sequences $(x_n)_{n=1}^{\infty}$, $x_n \in \mathbb{F}$

(3) $\ell^p, p = \infty$, the space of all $(x_n)_{n=1}^{\infty}$ with $\sup_{n\in\mathbb{N}}|x_n| < \infty$, i.e. the space of all bounded sequences

(4) $\ell^p, p \in [1, \infty[$, the space of all $(x_n)_{n=1}^{\infty}$ with $\sum_{n\in\mathbb{N}}|x_n|^p < \infty$

(5) $\ell^p, p \in [0, 1[$, the space of all $(x_n)_{n=1}^{\infty}$ with $\sum_{n\in\mathbb{N}}|x_n|^p < \infty$

(6) Space $C([0,1])$ of continuous functions on the interval $[0,1]$

(7) Solutions of Volterra's equation

(8) Space of polynomials $p \in X$ if $\exists n \in \mathbb{N} : p(x) = \sum_{j=0}^{n} a_j x^j$ ◊

Proof that (4) in example 2.2 is a linear space.
If $(x_n)_n$ with $\sum_n |x_n|^p < \infty$ and $(y_n)_n$ with $\sum_n |y_n|^p < \infty$, then $\sum_n |x_n + y_n|^p < \infty$?

$$|x_n + y_n|^p \leq |2x_n|^p + |2y_n|^p \leq 2^p(|x_n|^p + |y_n|^p).$$ ■

Remark 2.3 *(unit balls in ℓ^p)***.** Further investigations of ℓ^p-spaces: normable?

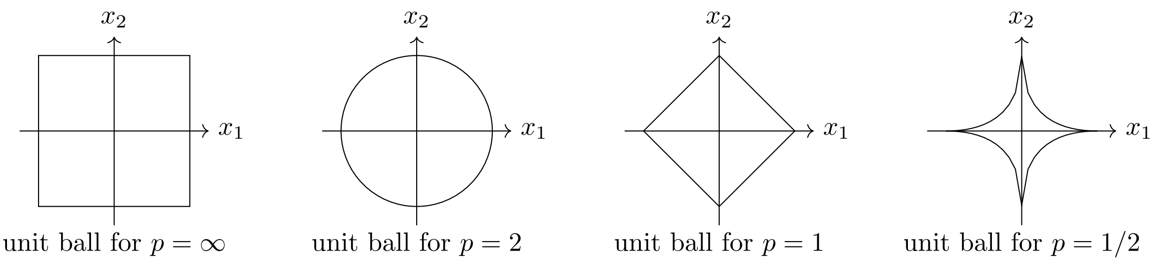

Figure 2: Unit balls in ℓ^p for some $p \in [0, \infty]$

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Definition 2.4 *(linear subspace)***.** $U \subseteq X$ is called *linear subspace* if $\forall x_1, x_2 \in U,\ \lambda_1, \lambda_2 \in \mathbb{F}:\ \lambda_1 x_1 + \lambda_2 x_2 \in U$.

Definition 2.5 *(sum of subsets in vector spaces)***.** If $S, T \subseteq X$ then $S + T := \{z \in X \mid z = x + y,\ x \in S,\ y \in T\}$.

Theorem 2.6 *(properties of linear subspaces)***.**

(1) $\{0\}$ and X are linear subspaces.

(2) The intersection of any collection of subspaces is a subspace.

(3) The sum of any collection of subspaces a subspace. □

Definition 2.7 *(linear span)***.** Given set $M \subseteq X$, the *linear span* $\operatorname{span}(M)$ is the intersection of all linear subspaces Y such that $M \subseteq Y$.

Theorem 2.8 *(properties of the linear span)***.**

(1) The linear span of M is the smallest linear subspace that includes M.

(2) $\operatorname{span}(M)$ consists precisely of the vectors $\sum_{j=1}^{n} \lambda_j x_j,\ n \in \mathbb{N},\ x_j \in M, \lambda_j \in \mathbb{F}$. □

Definition 2.9 *(convex set)***.** Only for $\mathbb{F} = \mathbb{R}$! K is *convex set* if for $x_1, x_2 \in K$ and $\lambda_1, \lambda_2 \in \mathbb{F},\ \lambda_1 + \lambda_2 = 1$ we have $\lambda_1 x_1 + \lambda_2 x_2 \in X$.

2.2 Normed Spaces

2015-04-21

Definition 2.10 *(normed space)***.** Let X be a linear space and $\|\cdot\| : X \to \mathbb{R}$ a map that satisfies:

(1)	non-negativity:	$\forall x \in X:$	$\|x\| \geq 0$
(2)	absolute homogenity:	$\forall x \in X, \lambda \in \mathbb{F}:$	$\|\lambda \cdot x\| = \lvert\lambda\rvert \cdot \|x\|$
(3)	triangle inequality:	$\forall x, y \in X:$	$\|x + y\| \leq \|x\| + \|y\|$
(4)	zero norm ⇒ zero vector:	$\forall x \in X:$	$\|x\| = 0 \ \Leftrightarrow\ x = 0$

Then $\|\cdot\|$ is called a *norm* on X, and $(X, \|\cdot\|)$ is called a *normed space*. On every normed space, we define a *distance function* d by:

$$d \colon X \times X \to \mathbb{R},\ d(x, y) = \|x - y\|.$$

Prop. 2.11 *(norms are Lipshitz continuous)***.** A norm $\|\cdot\| : X \to \mathbb{R}$ is uniformly continuous, and in fact even Lipshitz continuous. □

Proof. We have $\lvert\|x\| - \|y\|\rvert \leq \|x - y\|$. Put $y = -x + z$ into (3) to get $\lvert\|z\| - \|x\|\rvert \leq \|z - x\|$. ■

Definition 2.12 *(equivalence of norms)***.** Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X. They are called *equivalent* if

$$\exists C > 0 : \quad C^{-1} \cdot \|\cdot\|_2 \leq \|\cdot\|_1 \leq C^{+1} \cdot \|\cdot\|_2,$$

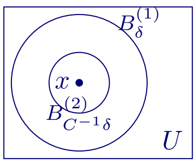
or equivalent to this condition,

$$\exists C, C' > 0 : \quad C' \cdot \|\cdot\|_2 \leq \|\cdot\|_1 \leq C \cdot \|\cdot\|_2.$$

Theorem 2.13 *(equivalence of norms)***.** Norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent iff the topologies they generate are the same. □

Proof.

Proof of "⇒": Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies. If $U \in \mathcal{T}_1$. $B_r^{(1)} := \{x \mid \|x\|_1 < r\}$. Then $B^{(2)}_{C^{-1}\delta} \subseteq B_\delta^{(1)} \subseteq B_{C\delta}^{(2)}$.



Proof of "⇐": $B_1^{(2)} \in \mathcal{T}_2$ if $T_1 = T_2$ therefore $B_C^{(1)} \supseteq B_1^{(2)}$. Let $x \in X$. Then $\frac{x}{\|x\|_2} \in \overline{B_1^{(2)}}$. With $B_C^{(1)} \supseteq B_1^{(2)}$ it follows that $\|\frac{x}{\|x\|_2}\|_1 \leq C$, and hence $\|x\|_1 \leq C\|x\|_2$.

■

Theorem 2.14 *(norms in finite-dim are equivalent)***.** All norms on a finite dimensional space are equivalent. □

Proof.

The one inequality. Let $\{e^1, \dots, e^n\}$ be a basis of X, so for any $x \in X$ we have $x = x_1 e^1 + \ldots + x_n e^n$. Consider the infinity-norm $\|x\|_\infty = \max_{1\leq j\leq n}|x_j|$. Let $\|\cdot\|$ be a different norm. Then

$$\begin{aligned}\|x\| &= \left\|x_1 e^1 + \ldots + x_n e^n\right\| \\ &\leq |x_1|\left\|e^1\right\| + \ldots + |x_n|\|e^n\| \\ &\leq \|x\|_\infty \cdot \underbrace{\left(\left\|e^1\right\| + \ldots + \|e^n\|\right)}_{=:C}.\end{aligned}$$

The other inequality. We observe that $\|\cdot\|$ is continuous in $\mathcal{T}_\infty$ (because $\|x\| \leq C \cdot \|x\|_\infty$). Let $S_1^\infty := \{x \mid \|x\|_\infty = 1\}$, then S_1^∞ is compact, and hence a minimum exists, $\min_{x\in S_1^\infty}\|x\| =: \delta > 0$ (where the latter inequality follows from $0 \notin S_1^\infty$). For any $x \in X$ we have $\frac{x}{\|x\|_\infty} \in S_1^\infty$, whereat

$$\left\|\frac{x}{\|x\|_\infty}\right\| \geq \delta \quad \therefore \quad \|x\| \geq \delta\|x\|_\infty.$$ ■

Theorem 2.15 *(compactness of the closed unit ball)***.** Closed unit ball $\overline{B_1} := \{x \in X \mid \|x\| \leq 1\}$ is compact iff dimension of X is finite. □

Proof of theorem 2.15 – part 1/2. If X is infinite-dimensional, then $\overline{B_1}$ is not compact. ■

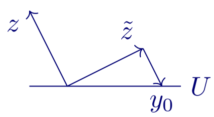
Example 2.16. $(\ell^\infty, \|\cdot\|_\infty)$, i.e. all bounded sequences, where $\|x\|_\infty = \sup_{j\in\mathbb{N}}|x_j|$. Then $\forall j : \|e^j\|_\infty = 1$ and $\forall j \neq k$: $\|e^j - e^k\|_\infty = 1$, where

$$\begin{aligned}e^1 &= (1, 0, 0, 0, \dots) \\ e^2 &= (0, 1, 0, 0, \dots) \\ e^3 &= (0, 0, 1, 0, \dots).\end{aligned}$$

In particular $e^1, e^2, e^3, \dots$ is neither convergent nor Cauchy. ◊

Lemma 2.17 *(existence of projections)***.** Let U be a proper closed linear subspace of X. Then there exists $x \notin U$ with $\|x\| = 1$ such that $\operatorname{dist}(x, U) \geq \frac{1}{2}$, where $\operatorname{dist}(x, U) = \inf_{y\in U}\|x - y\|$. □

Proof.

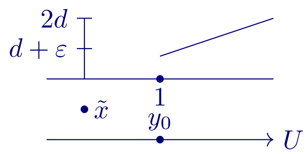


Pick any $\tilde{x} \notin U$, then $\operatorname{dist}(\tilde{x}, U) = d > 0$ (because U is closed). Pick $y_0 \in U$ such that $\|\tilde{x} - y_0\| = 2d$. Claim ist that $x := \frac{\tilde{x}-y_0}{2d}$ satisfies the requirements. Clearly $\|x\| = 1$. Let $y \in U$. Then

$$\|x - y\| = \left\|\frac{\tilde{x} - y_0}{2d} - y\right\| = \left\|\frac{\tilde{x} - y_0 - 2dy}{2d}\right\| \geq \frac{d}{2d} = \frac{1}{2}.$$

Since U is linear subspace $y_0 + 2dy \in U$, and hence $\overset{\|\tilde{x}-z\|2d}{\geq} \frac{d}{2d} = \frac{1}{2}$. ■

Remark 2.18.



Concering the $\operatorname{dist}(x, U) = \inf_{y\in U}\|x - y\|$: There exists a sequence $y_n \in U$ such that $\|y_n - x\| \overset{n\to\infty}{\longrightarrow} d$, in particular for any $\varepsilon > 0$ there is a $y(\varepsilon)$ such that $\|y(\varepsilon) - x\| \leq d + \varepsilon$. If instead of $y(\varepsilon)$ you consider $\lambda y(\varepsilon)$, $\varepsilon\mathbb{R}$.

$$F(\lambda) := \|\lambda y(\varepsilon) - x\|, \ \lambda \in \mathbb{R}$$

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Proof of theorem 2.15 – part 2/2. If X is infinite-dimensional, then $\overline{B_n}$ is not compact. We construct a sequence $(x_0, x_1, \dots, x_n, \dots)$, $x_j \in X$ where x_0 is arbitrary with $\|x_0\| = 1$. Given $(x_0, \dots, x_n)$ then consider $\operatorname{span}\{x_1, \dots, x_n\} =: U$ (closed because of finite dimensional, and hence proper). Use the lemma to pick x_{n+1} such that $\forall j : \|x_j\| = 1$ and $\forall j \neq k : \|x_j - x_k\| \geq \frac{1}{2}$. ■

Remark 2.19. We have a look at subspaces of $(c, \|\cdot\|)$:

$$\|x\| = \max_{n\in\mathbb{N}}|x_n|$$
$$c_0 = \left\{\text{infinite real sequences } (x_n)_{n\in\mathbb{N}} \;\middle|\; \lim_{n\to\infty} x_n = 0\right\}$$
$$c_{\text{cpt}} = \{\text{sequences } (x_n)_{n\in\mathbb{N}} \mid (x_n)_{n\in\mathbb{N}} \text{ has only finitely many non-zero elements}\}$$
$$c_{\text{cpt}} \text{ is a proper subspace of } c$$ //

Repitition:
- equivalent norms
- topologies
- finite-dimensional ⇔ all norms equivalent
- unit ball is not compact in infinite-dimensional spaces

Question: Suppose you have two topologies $\mathcal{T}_1, \mathcal{T}_2$ induces by norms $\|\cdot\|_1, \|\cdot\|_2$. ...

2.3 Banach Spaces

Definition 2.20 *(Banach space)***.** Banach space is a normed linear space that is complete.

Motivation: Why Banach?
- numerical analysis: $\lim_{n\to\infty} x_n, \quad |x_n - x_k| < \text{precision}, \; n,k \geq n_0$
- pure math: $x_{n+1} = F(x_n, x_{n-1}), \quad \lim_{n\to\infty} x_n = x \;\Leftrightarrow\; x = F(x,x)$

Example 2.21 *(examples and counterexamples for banach spaces)***.**

(1) c, the space of real/complex sequences $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n$ exists. Equipped with norm $\|(x_n)_n\| = \max_{n\in\mathbb{N}}|x_n|$ it is Banach.

(2) c_0, the space $c_0 \subseteq c$ of sequences such that $\lim_{n\to\infty} x_n = 0$. This is a closed subspace, hence a Banach space.

(3) c_{cpt}, the space $c_{\text{cpt}} \subseteq c_0$ of sequences with finite number of non-zero elements.
Claim. c_{cpt} is a proper dense subspace of c_0.
Proof.
Proper: $x_n = \frac{1}{n}$.
Dense: Let $(x_n)_n \in c_0$ and pick ε. Find N such that $|x_n| \leq \varepsilon$ for $n \geq N$. Define $x_n^{(N)} = \left\{\begin{smallmatrix} x_n & \text{for } n\leq N \\ 0 & \text{for } n>N \end{smallmatrix}\right.$. Clearly $(x_n^{(N)})_n \in c_{\text{cpt}}$. Furthermore $\|(x_n^{(N)})_n - (x_n)_n\| = \max_n|x_n^{(N)} - x_n| = \max_{n\geq N}|x_n| \leq \varepsilon$.

(4) Let (M,d) be a metric space and $K \subseteq M$ be a compact set.
$C(K)$, the space of all continuous functions $f\colon K \to \mathbb{R}$.
Norm on this space: $\|f\|_\infty = \sup_{x\in K}|f(x)|$ (called the *max-norm* or *sup-norm*) ◊

Concering the fourth example:
Question: If $f_n \in C(K)$ such that $\forall x \in K: \; \lim_{n\to\infty} f_n(x) = f(x)$, does it imply that $f \in C(K)$?
Negative answer: No!, take $f_n = x^n$.
Positive answer: Yes!, if $f_n \rightrightarrows f$. Recall:

$$\begin{aligned}
f_n \to f \quad &\Leftrightarrow \quad \forall x: \forall\varepsilon > 0\; \exists n_0 \in \mathbb{N}\; \forall n \geq n_0 : |f_n(x) - f(x)| \leq \varepsilon \\
f_n \rightrightarrows f \quad &\Leftrightarrow \quad \forall\varepsilon > 0\; \exists n_0 \in \mathbb{N}\; \forall n \geq n_0 : \forall x: \; |f_n(x) - f(x)| \leq \varepsilon \\
&\Leftrightarrow \quad \forall\varepsilon > 0\; \exists n_0 \in \mathbb{N}\; \forall n \geq n_0 : \; \max_{x\in K}|f_n(x) - f(x)| \leq \varepsilon \\
&\Leftrightarrow \quad \forall\varepsilon > 0\; \exists n_0 \in \mathbb{N}\; \forall n \geq n_0 : \|f_n - f\|_\infty \leq \varepsilon \\
&\Leftrightarrow \quad \|f_n - f\|_\infty \overset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Remark 2.22 *(convergence in sup-Norm = uniform convergence)***.** Notion if convergence w.r.t. the norm $\|\cdot\|_\infty$ is equivalent to the notation of uniform convergence. //

Theorem 2.23 *($(C(K), \|\cdot\|_\infty)$ is complete)***.** $(C(K), \|\cdot\|_\infty)$ is a Banach space. □

Proof. If $f_n \in C(K)$ Cauchy sequence, $\|f_n - f_k\|_\infty = \max_{x\in K}|f_n(x) - f_k(x)| \le \varepsilon$ if $n, k \ge N$, then $f_n \longrightarrow f$. For each $x \in K$, then $f_n(x)$ is a Cauchy sequence, then $f(x) := \lim_{n\to\infty} f_n(x)$ exists.
To show:
 (a) $\|f_n - f\|_\infty \longrightarrow 0$
 (b) $f \in C(K)$
Proof:
 (a) Pick N from above. Then

$$\begin{aligned}\|f - f_N\|_\infty &= \max_{x\in K}|f(x) - f_N(x)| \\ &= \max_{x\in K}\lim_{n\to\infty}|f_n(x) - f_N(x)| \\ &\le \sup_{x\in K}\sup_{n\ge N}|f_n(x) - f_N(x)| \\ &\le \sup_{n\ge N}\sup_{x\in K}|f_n(x) - f_N(x)| \\ &\le \varepsilon.\end{aligned}$$

 (b) Fix N such that $|f(x) - f_N(x)| \le \frac{\varepsilon}{3}$ and $|f(y) - f_N(y)| \le \frac{\varepsilon}{3}$. Now since f_N continuous choose x, y such that $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$ if $d(x, y) < \delta$. Then

$$|f(x) - f(y)| \le \underbrace{|f(x) - f_N(x)|}_{\le\varepsilon/3} + \underbrace{|f_N(x) - f_N(y)|}_{\le\varepsilon/3} + \underbrace{|f(y) - f_N(y)|}_{\le\varepsilon/3} \le \varepsilon.$$ ■

What are compact subsets of $C(K)$?

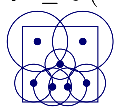
Prop. 2.24 *(characterization of relative compactness)***.** The following is equivalent for subsets N of complete metric spaces:

(i) $\overline{N}$ compact

(ii) Every sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in N$ has a convergent subsequence

(iii) For each $\varepsilon > 0$ exists a finite number of $x_j \in N$, $j = 1, \ldots, n$ such that $\bigcup_{j=1,\ldots,n} B_\varepsilon(x_j) = N$ □

Remark 2.25 *(prequesits for Arzela-Ascoli)***.** Let K be a compact set, and consider $(C(K), \|\cdot\|_\infty)$, and let $\mathcal{F} \subseteq C(K)$.
Recall that:



 – $\mathcal{F}$ is *bounded* if $\sup_{f\in\mathcal{F}}\|f\|_\infty < \infty$.
 – $\mathcal{F}$ is called *equicontinuous* if

$$\forall x: \ \forall\varepsilon > 0\ \exists\delta > 0\ \forall f \in \mathcal{F}: \ \forall y: \ d(x, y) \le \delta \ \Rightarrow \ |f(x) - f(y)| \le \varepsilon.$$

//

2015-04-28

> *Repitition:*
> – (relative) compactness
> – Arzela-Ascoli theorem
> – equicontinuity

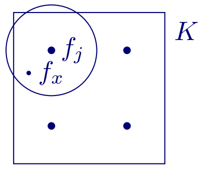
Prop. 2.26 *(continuous functions map compact sets to compact sets)***.** Continuous functions map compact sets to compact sets. In particular, continuous function on a compact set attains its maxima/minima. □

> *Motivation:* Problem: Given function $f: K \to \mathbb{R}$, find $\min_{x\in K} f(x)$. $\to$ find a topology, that has so much open sets such that f is continuous, but so less open sets, such that K is compact.

Remark 2.27. Every finite set of continuous functions is equicontinuous. //

Theorem 2.28 *(Arzela-Ascoli)***.** Let K be a compact set, and consider $(C(K), \|\cdot\|_\infty)$, and let $\mathcal{F} \subseteq C(K)$. Then $\mathcal{F}$ is relatively compact, iff $\mathcal{F}$ is equicontinuous and bounded. □

Proof of $\mathcal{F}$ relatively compact $\Rightarrow$ $\mathcal{F}$ equicontinuous & bounded.



For any ε there are functions $\{f_j\}_{j=1}^{N(\varepsilon)}$ such that:

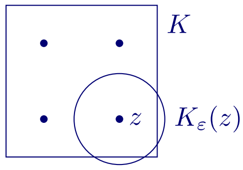
$$\bigcup_{j=1}^{N(\varepsilon)} B_\varepsilon(f_j) \supseteq \mathcal{F}.$$

Let $f \in \mathcal{F}$ and pick x, then

$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f(y) - f_j(y)| + |f_j(x) - f_j(y)| \leq 3\varepsilon,$$

where pick a j such that $\|f - f_j\| \leq \varepsilon$, and a δ such that $d(x, y) \leq \delta \Rightarrow |f_j(x) - f_j(y)| \leq \varepsilon$. Idea for bounded is similar. ■

Proof of $\mathcal{F}$ relatively compact $\Leftarrow$ $\mathcal{F}$ equicontinuous & bounded.



We need to prove that given $f_n \in \mathcal{F}$, then there is a subsequence $f_{n(j)}$ such that $\lim_{j\to\infty} f_{n(j)}$ exists.

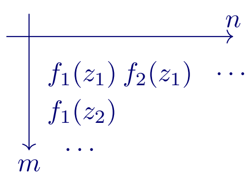
- $\exists f \in C(K)$: $\|f - f_{n(j)}\| \longrightarrow 0$
- For j, k large $\|f_{n(k)} - f_{n(j)}\| \longrightarrow 0$ meaning that subsequence $f_{n(j)}$ is cauchy.

Steps (overview):

1. Find the covering $K \subseteq \bigcup_{z\in S} K_r(z)$, i.e. construct such $K_r(z)$'s and S.
2. Diagonal trick:
   Consider $f_n(z)$ for $z \in S$. Then there is a $n(j)$ such that $f_{n(j)}(z)$ converges for all $z \in S$ (use boundness).
3. Use construction of S to prove that $f_{n(j)}(z)$ is Cauchy (use equicontinuity).

Steps (details):

1. Construction of $K_\varepsilon(z)$'s:
   For each $\varepsilon > 0$ and $z \in K$ define
   $$K_\varepsilon(z) = \{x \in K \mid \forall f \in \mathcal{F} |f(z) - f(x)| \leq \varepsilon\}.$$
   Because $\mathcal{F}$ is equicontinuous, $K_\varepsilon(z)$ is nonempty and open, and $K \subseteq \bigcup_{z\in K} K_\varepsilon(z)$.
   Construction of S:
   Pick N such that $K \subseteq \bigcup_{z\in K} K_{1/N}(z)$. Choose $K_N \subseteq K$ such that $K_N = \{z_1, \dots, z_n\}$ discrete set and
   $$K \subseteq \bigcup_{z\in K} K_{1/N}(z) \subseteq \bigcup_{z\in K_N} K_{1/N}(z).$$
   Define $S := \bigcup_{N\in\mathbb{N}} K_N$, then S is countable.
3. Claim: $f_{n(j)}$ constructed in step 2 is a Cauchy sequence.
   Proof: For all $x \in K$ and $z \in S$ it holds that
   $$\left|f_{n(j)}(x) - f_{n(k)}(x)\right| \leq \left|f_{n(j)}(x) - f_{n(j)}(z)\right| + \left|f_{n(k)}(x) - f_{n(k)}(z)\right| + \left|f_{n(j)}(z) - f_{n(k)}(z)\right|.$$
   Pick $N > 0$ and $z \in K_N$ such that $|f_{n(j)}(x) - f_{n(j)}(z)| \leq \frac{1}{N}$ for all j. Pick j, k such that $|f_{n(j)}(z) - f_{n(k)}(z)| \leq \frac{1}{N}$.
   Then for all x there exists N, n_0 such that
   $$j, k \geq n_0 \;\Rightarrow\; \left|f_{n(j)}(x) - f_{n(k)}(x)\right| \leq \tfrac{3}{N},$$
   and hence $\|f_{n(j)} - f_{n(k)}\| \leq \frac{3}{N}$.
2. *Lemma (diagonal trick).* Let S be a countable set and let $f_n(z), n \in \mathbb{N}$ be a sequence such that there is a $M > 0$ with $\forall n \in \mathbb{N}, z \in S : |f_n(z)| \leq M$. Then there exists a subsequence $n(j)$ such that $f_{n(j)}(z)$ is convergent for all $z \in S$.

   Proof. Since S is countable, $S = \{z_1, z_2, \dots\} = \{z_m\}_{m\in\mathbb{N}}$. Then we have sequences $f_n(z_m)$. Because the sequence $\{f_n(z_1)\}_{n\in\mathbb{N}}$ is bounded, there is a subsequence $n_1(j)$ such that $f_{n_1(j)}(z_1)$ is convergent, and there is a subsequence $n_2(j)$ of $n_1(j)$ such that $f_{n_2(j)}(z_2)$ is convergent, and so on. Continuing this process, you can find subsequence $n_m(j)$ such that $f_{n_m(j)(z_k)}$ converges for $k \leq m$.
   Naive: Define $n_\infty(j) := \lim_{m\to\infty} n_m(j)$. It may happen that $\lim_{m\to\infty} n_m(1) = \infty$.
   Correct: Pick a subsequence $n_\infty(j) := n_j(j)$. Claim is that $f_{n_j(j)}(z)$ is convergent for all $z \in S$.
   Proof: Pick any z, let say $z = z_{100}$, then $f_{n_j(j)}$ is convergent, $n_{100}(j)$ is a subsequence for which $f_{n_{100}(j)}(z_{100})$ is convergent.

This finishes the proof. ■

2.4 Inner Product Spaces

Definition 2.29 *(inner product space)*. Let V be a linear space and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ a map that satisfies

(1)	non-negativity:	$\forall x \in V:$	$\langle x, x\rangle \geq 0$
(2)	linear in 2$^{\text{nd}}$ argument:	$\forall x, y \in V, \lambda \in \mathbb{C}:$	$\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$
(3)	linear in 2$^{\text{nd}}$ argument:	$\forall x, y, z \in V:$	$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$
(4)	hermitian:	$\forall x, y \in V:$	$\langle x, y\rangle = \overline{\langle y, x\rangle}$
(5)	definiteness:	$\forall x \in V:$	$\langle x, x\rangle = 0 \Leftrightarrow x = 0$

Then $\langle\cdot,\cdot\rangle$ is called a *scalar product* in V, and $(V, \langle\cdot,\cdot\rangle)$ is called a *normed space*. We claim:

(2')	semilinear in 1$^{\text{st}}$ argument:	$\forall x, y \in V, \lambda \in \mathbb{C}:$	$\langle \alpha x, y\rangle = \overline{\alpha}\langle x, y\rangle$
(3')	semilinear in 1$^{\text{st}}$ argument:	$\forall x, y, z \in V:$	$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

Furthermore, the scalarproduct $\langle\cdot,\cdot\rangle$ induces a norm $\|\cdot\|$ by

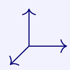
$$\|\cdot\| : V \to \mathbb{R}, \ \|x\| := \sqrt{\langle x, x\rangle}.$$

Example 2.30 *(examples of inner product spaces)*.

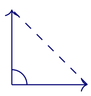
(1) $\mathbb{C}^n$ equipped with $\langle x, y\rangle = \sum_{j=1}^n \overline{x_j} \cdot y_j$ is an inner product space, and a Banach space.

(2) $C([0,1])$ equipped with $\langle f, g\rangle = \int_0^1 \overline{f(x)} \cdot g(x)\,\mathrm{d}x$ is an inner product space, but not a Banach space. ◊

Definition 2.31 *(orthogonality)*.
Vectors x, y are orthogonal, $x \perp y$, if $\langle x, y\rangle = 0$. A set of vectors $\{x_j\}_{j\in J}$ is called an orthonormal set, if they are mutually orthogonal and $\forall j \in J : \|x_j\| = 1$.



The Pythagorean theorem states that, if $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. We generalize this statement.



Theorem 2.32 *(Pythagoras theorem)*. Let $\{x_j\}_{j=1}^\infty$ be an orthonormal set and $x \in V$. Then

$$\|x\|^2 = \sum_{j=1}^n |\langle x_j, x\rangle|^2 + \left\| x - \sum_{j=1}^n x_j \langle x_j, x\rangle \right\|^2.$$

□

Proof. Notice that $\left(x - \sum_{j=1}^n x_j\langle x_j, x\rangle\right) \perp x_k$:

$$\left\langle x_k, x - \sum_{j=1}^n x_j \langle x_j, x\rangle \right\rangle = \langle x_k, x\rangle - \langle x_k, x\rangle = 0$$

Then use pythogorean relation $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ repeatly:

$$\begin{aligned} x &= \left(x - \sum_{j=1}^n x_j\langle x_j, x\rangle\right) + \left(\sum_{j=1}^n x_j\langle x_j, x\rangle\right) \\ \|x\|^2 &= \left\| x - \sum_{j=1}^n x_j\langle x_j, x\rangle \right\|^2 + \left\| \sum_{j=1}^n x_j\langle x_j, x\rangle \right\|^2 \\ &= \left\| x - \sum_{j=1}^n x_j\langle x_j, x\rangle \right\|^2 + \left\| x_1\langle x_1, x\rangle + \sum_{j=2}^n x_j\langle x_j, x\rangle \right\|^2 \\ &= \left\| x - \sum_{j=1}^n x_j\langle x_j, x\rangle \right\|^2 + |\langle x_1, x\rangle|^2 + \left\| \sum_{j=2}^n x_j\langle x_j, x\rangle \right\|^2 \end{aligned}$$

■

Corollary 2.33 *(Bessel inequality)*. For any orthonormal set $\{x_j\}_{j=1}^n$ and vector $x \in V$, we so-called *Bessel inequality* holds, that is

$$\|x\|^2 \geq \sum_{j=1}^{n} |\langle x_j, x\rangle|^2. \qquad \square$$

Corollary 2.34 *(Cauchy-Schwarz inequality)*. For all $x, y \in V$ it holds that

$$\|x\| \cdot \|y\| \geq |\langle x, y\rangle|. \qquad \square$$

Proof of Cauchy-Schwarz – using Bessel inequality. For any $y \neq 0$ $\{\frac{y}{\|y\|}\}$ is an orthonormal set. Bessel inequality implies

$$\|x\|^2 \geq \left|\left\langle \frac{y}{\|y\|}, x\right\rangle\right|^2 = \frac{|\langle x, y\rangle|^2}{\|y\|^2}. \qquad \blacksquare$$

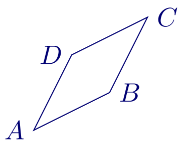
Proof of Cauchy-Schwartz – typical proof. Suppose $\langle x, y\rangle \in \mathbb{R}$. Then for all $t \in \mathbb{R}$ we have that

$$0 \leq \langle x - ty, x - ty\rangle = \|x\|^2 - 2t\langle x, y\rangle + t^2\|y\|^2.$$

This expression is minimal at $t = \frac{\langle x,y\rangle}{\|y\|^2}$, and so

$$0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}. \qquad \blacksquare$$

Every parallelogram, e.g. the one drawn on the righthand side, satisfies the identity



$$|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = |AC|^2 + |BD|^2.$$

We transfer this identity to *normed* spaces (where it doesn't have to be true, cf. proposition 2.35), and call it *parallelogram identity*:

$$\forall x, y \in V: \ \|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Prop. 2.35 *(characterization of inner product spaces)*. Norm is associated to a scalar product, iff the parallelogram identity holds. □

2.5 Hilbert Spaces

Definition 2.36 *(Hilbert space)*. An inner product space complete in this norm is called a *Hilbert space*.

Example 2.37 *(examples of Hilbert spaces)*.

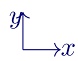
(3) $L^2([0, 1])$ of functions with $\int_0^1 |f(x)|^2 \,\mathrm{d}x < 0$, equipped with $\langle f, g\rangle := \int_0^1 \overline{f(x)} \cdot g(x) \,\mathrm{d}x$ is a Hilbert space.

(4) ℓ^2 of sequences with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, equipped with $\langle x, y\rangle := \sum_{n=1}^{\infty} \overline{x_n} \cdot y_n$ is a Hilbert space. ◊

Remark 2.38. No other ℓ^p spaces, except for ℓ^2, are Hilbert spaces. //

Prop. 2.39 *(product of Hilbert spaces)*. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. Then $\mathcal{H}_1 \times \mathcal{H}_2 := \{(x, y) \mid x \in \mathcal{H}_1, y \in \mathcal{H}_2\}$ is a Hilbert space with inner product $\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle_{\mathcal{H}_1} + \langle y_1, y_2\rangle_{\mathcal{H}_2}$. □

Remark 2.40. Preview: Decomposition of Hilbert spaces: "$\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$". //



Definition 2.41 *(orthogonal complement)*. Let U be a linear subspace of $\mathcal{H}$. Then $U^\perp := \{x \in \mathcal{H} \mid \forall y \in U : x \perp y\}$

Lemma 2.42 *(properties of the orthogonal complement)*. $U^\perp$ is linear subspace, and in fact it is a closed subspace. □

Proof. Closed: Exercise. Linear: If $y_1, y_2 \in U^\perp$, then also $\alpha y_1 + \beta y_2 \in U^\perp$. Pick $x \in U$, we need to prove

$$\langle x, \alpha y_1 + \beta y_2\rangle = \alpha\langle x, y_1\rangle + \beta\langle x, y_2\rangle = 0.$$

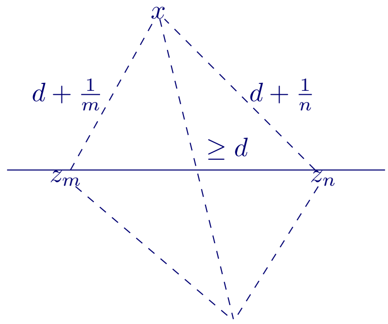
■

Lemma 2.43 *(existence of projections)***.** Let U be a closed proper linear subspace of $\mathcal{H}$ (Hilbert space), and $x \in \mathcal{H}$. Then there exists a unique $z \in U$ that minimizes $\|x - y\|$ for $y \in U$, i.e.

$$\operatorname{dist}(x, U) := \inf_{y \in U} \|x - y\| = \|x - z\|. \qquad \square$$

Proof. Let $d := \inf_{y \in U} \|x - y\|$. And let z_n be a minimizing sequence, i.e. $\|x - z_n\| \overset{n\to\infty}{\longrightarrow} d$, for example $\|x - z_n\|^2 = d^2 + \frac{1}{n}$.

We are going to show that $(z_n)_{n\in\mathbb{N}}$ is Cauchy.



$$\begin{aligned}
\|z_n - z_m\|^2 &= \|(x - z_m) - (x - z_n)\|^2 \\
&= 2\left(\|x - z_n\|^2 + \|x - z_m\|^2\right) - \|2x - z_n - z_m\|^2 \\
&= 4d^2 + 2\left(\tfrac{1}{n} + \tfrac{1}{m}\right) - 4\left\|x - \tfrac{1}{2}(z_n + z_m)\right\|^2 \\
&\leq 4d^2 + 2\left(\tfrac{1}{n} + \tfrac{1}{m}\right) - 4d^2 \\
&= 2\left(\tfrac{1}{n} + \tfrac{1}{m}\right)
\end{aligned}$$

Therefore the sequence is Cauchy.

Existance is done, now uniqueness. Let z and $\tilde{z}$ be two minimizers, $\|x - z\| = \|x - \tilde{z}\| = d$. Use parallelogram identity on $x - z$ and $x - \tilde{z}$ yields $\|z - \tilde{z}\| \leq 0$. ■

2015-05-08

> *Repitition:*
>
> - Inner product spaces, Hilbert spaces
> - Bessel inequality, Pythogoras theorem
> - orthogonal complement
> - existence of projection

Lemma 2.44 *(existence of projections – convex version)***.** Let K be a closed convex set, $K \subseteq \mathcal{H}$. Then for each $x \in \mathcal{H}$, there exists a unique $y \in K$ that minimizes the distance of x to K. □

Proof. Similar to proof of the same for linear subset K. ■

Example 2.45 *(existence of projections – counterexample)***.**
Lemma 2.44 is not true if we consider non-convex spaces.

◊

Lemma 2.46 *(projection lemma)***.** Let $U \subseteq \mathcal{H}$ be a closed linear subspace. Then each point $x \in \mathcal{H}$ has a unique decompositon $x = z + w$ where $z \in U$ and $w \in U^\perp$. □

Proof. Let $x \in \mathcal{H}$, then there exists a $z \in U$, such that $\operatorname{dist}(x, U) = \|z - x\|$. We have z, and put $w = x - z$. Claim $w \in U^\perp$. We know that for each $y \in U$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\|x - z\|^2 &\leq \|x - z - \alpha y\|^2 \\
&= \langle x - z - \alpha y, x - z - \alpha y\rangle \\
&= \|x - z\|^2 - \langle x - z, \alpha y\rangle - \langle \alpha y, x - z\rangle + \langle \alpha y, \alpha y\rangle \\
&= \|x - z\|^2 - \alpha\langle x - z, y\rangle - \overline{\alpha}\overline{\langle x - z, y\rangle} + |\alpha|^2\|y\|^2
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\forall y \in U \ \forall \alpha \in \mathbb{C}: \qquad 0 \leq |\alpha|^2\|y\|^2 - \alpha\langle x - z, y\rangle - \overline{\alpha}\overline{\langle x - z, y\rangle} \\
&\forall y \in U \ \forall \alpha = r \in \mathbb{R}: 0 \leq t^2\|y\|^2 - 2t\operatorname{Re}\langle x - z, y\rangle
\end{aligned}$$

Therefore $\operatorname{Re}\langle x - z, y\rangle = 0$, and with $\alpha = it$ it follows that $\langle w, y\rangle = \langle x - z, y\rangle = 0$, and hence $w \in U^\perp$. ■

Prop. 2.47. For every closed linear subspace $U \subseteq \mathcal{H}$, the dirct sum $U \oplus U^\perp$ is isometric to $\mathcal{H}$, and an isometry is given by $(z, w) \mapsto z + w$. □

Proof. $f(\alpha) = \|x - z - \alpha y\|^2$, $f'(0) = 0$. ■

2.6 The Dual Space to a Hilbert Space

Definition 2.48 *(dual space)***.** A map $\varphi\colon \mathcal{H} \to \mathbb{C}$ is called a *linear functional*, if it is a bounded linear map, i.e.:
(1) Linearity: $\forall x, y \in \mathcal{H},\ \alpha \in \mathbb{C}:\ \varphi(x + \alpha y) = \varphi(x) + \alpha\varphi(y)$
(2) Boundedness: $\exists C \in \mathbb{R}:\ |\varphi(x)| \le C\|x\|_{\mathcal{H}}$
The space of all linear functionals on $\mathcal{H}$ is called the *dual space* $\mathcal{H}^*$ of $\mathcal{H}$. We equip $\mathcal{H}^*$ with a norm $\|\cdot\|_{\mathcal{H}^*}$,

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{x\in\mathcal{H}, \|x\|=1} |\varphi(x)| = \sup_{x\in\mathcal{H}, x\neq 0} \frac{|\varphi(x)|}{\|x\|}.$$

Remark 2.49. *Remark by the typesetter: This definition holds for any normed space, not just Hilbert spaces. Anyway, the more general definition will come in definition 3.6. Furhtermore, the norm $\|\cdot\|_{M^*}$ conincides with the operator norm $\|\cdot\|_{M\to\mathbb{F}}$.* //

Remark 2.50 *(kernel of linear functional is a hyperplane)***.** Hyperplanes in $\mathbb{R}^n$ can be denoted by $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ where $a_j \in \mathbb{R}$. Given any φ, the solution of $\varphi(x) = 0$ forms a hyperplane. //

Prop. 2.51 *(properties of dual spaces)***.** If $\mathcal{H}$ is a Hilbert space, then $\mathcal{H}^*$ is a Banach space, and in fact it is a Hilbert space. □

Example 2.52 *(examples for dual spaces)***.**

(1) For $\mathcal{H} = L^2([0,1])$, for any $g \in L^2([0,1])$, $\varphi(f) = \int_0^1 g(x)f(x)\,\mathrm{d}x$. ◊

We generalize example 2.52 to arbitrary Hilbert spaces.
Lemma 2.53 *(every vector induces a linear functional)***.** Let $\mathcal{H}$ be an arbitrary Hilbert space. Then any $y \in \mathcal{H}$ induces a linear function φ_y by $\varphi_y(x) = \langle y, x\rangle$. □

Proof. Bounded because of Cauchy-Schwarz,

$$|\varphi_y(x)| = |\langle y, x\rangle| \le \|y\|\,\|x\| \quad \therefore \quad \sup_{\|x\|=1} |\varphi_y(x)| \le \|y\|.$$

Other way to see boundness:

$$\mathcal{N} := \ker\varphi_y(x) := \{x \in \mathcal{H} \mid \varphi_y(x) = 0\} = \operatorname{span}(y)^\perp.$$

Because $\mathcal{H} = \mathcal{N} + \mathcal{N}^\perp$, we can decompose any $x \in \mathcal{H}$ into $x = \alpha y + w$.

$$\varphi_y(x) = \langle y, \alpha y + w\rangle = \alpha\|y\|^2$$
$$\|x\|^2 = |\alpha|^2\|y\|^2 + \|w\|^2$$

$w = 0$ and $\alpha = \frac{1}{\|y\|}$ implies $\|x\| = 1$.

$$\varphi_y(x) = \tfrac{1}{\|y\|}\|y\|^2 = \|y\|$$
$$\sup_{\|x\|=1} \varphi_y(x) \ge \varphi_y(x) = \|y\|$$ ■

Theorem 2.54 *(every linear functional is induced by a vector = Riesz representation theorem)***.** Let $\varphi \in \mathcal{H}^*$. Then there is a unique $y_\varphi \in \mathcal{H}$ such that $\forall x \in \mathcal{H} : \varphi(x) = \langle y_\varphi, x\rangle$. Furthermore, $\|\varphi\|_{\mathcal{H}^*} = \|y_\varphi\|_{\mathcal{H}}$. □

Proof. Let $N = \ker\varphi = \{x \in \mathcal{H} \mid \varphi(x) = 0\}$. Then $\mathcal{N}$ is closed linear subspace (closed follows from boundness of φ, more explicit proof later). If $\mathcal{N} = \mathcal{H}$ then $\varphi = 0$ and $y_\varphi = 0$. Suppose that $\mathcal{N} \neq \mathcal{H}$. It follows by the projection lemma that there exists a $w_0 \in \mathcal{N}^\perp$, then we can write a decomposition,

$$x = \underbrace{\left(x - \frac{\varphi(x)}{\varphi(w_0)}w_0\right)}_{=:y\in\mathcal{N}} + \underbrace{\frac{\varphi(x)}{\varphi(w_0)}w_0}_{\in\mathcal{N}^\perp},$$

where $y \in \mathcal{N}$ follows by

$$\varphi(y) = \varphi\left(x - \frac{\varphi(x)}{\varphi(w_0)}w_0\right) = \varphi(x) - \varphi(x) = 0.$$

All functionals $\alpha\langle w_0, x\rangle, \alpha \in \mathbb{C}$. We need to just find the $\alpha \in \mathbb{C}$ such that $\varphi(w_0) = \alpha\langle w_0, w_0\rangle$. Hence $\alpha = \frac{\varphi(w_0)}{\|w_0\|^2}$. Claim is that $\varphi_y(x) = \langle \frac{\varphi(w_0)}{\|w_0\|^2} w_0, x\rangle$, i.e. $y_\varphi = \frac{\varphi(w_0)}{\|w_0\|^2} w_0$.

Uniqueness: Suppose we have y_φ and $\tilde{y}_\varphi$ that satisfy the lemma. Then $\forall x \in \mathcal{H} : \langle y_\varphi - \tilde{y}_\varphi, x\rangle = 0$, in particular $x = y_\varphi - \tilde{y}_\varphi$, therefore $\|y_\varphi - \tilde{y}_\varphi\|^2 = 0$, and hence $y_\varphi = \tilde{y}_\varphi$. ■

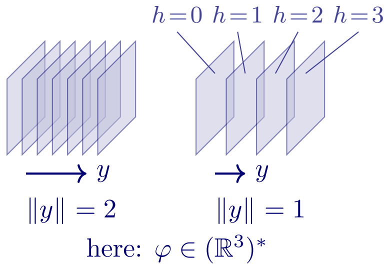
Corollary 2.55 *(norm of induced functional)***.** In particular it follows from theorem 2.54 that

$$\forall y \in \mathcal{H} : \|\varphi_y\|_{\mathcal{H}^*} = \|y\|_{\mathcal{H}} \quad \text{and} \quad \forall \varphi \in \mathcal{H}^* : \|\varphi\|_{\mathcal{H}^*} = \|y_\varphi\|_{\mathcal{H}}.$$ □

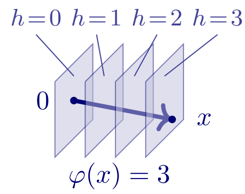
Corollary 2.56 *($\mathcal{H}^*$ is isomorphic to $\mathcal{H}$)***.** $\mathcal{H}^*$ is isomorphic to $\mathcal{H}$: By lemma 2.53 and theorem 2.54 every vector $y \in \mathcal{H}$ corresponds to a linear functional $\varphi \in \mathcal{H}^*$ (via $y \mapsto \varphi_y$), and vice versa. Furthermore, by corollary 2.55 this bijection ($y \mapsto \varphi_y$) is isometric. □

Remark 2.57 *(visualization of linear functionals in finite dimensions)***.** *Remark by the typesetter: This remark is written by the typesetter of the script, and is not part of the lecture itself, but it extends remark 2.50.*

For the sake of imagination, we consider the Hilbert space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. Let $\varphi \in (\mathbb{R}^n)^*$ be a linear functional. The level sets of $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ are parallel hyperplanes. If we choose the levels to be equidistant (e.g. $0, 1, 2, \dots$), then the levels sets are equidistant too. We can also think of these hyperplanes as wave fronts of a plane wave. By virtue of the *Riesz representation theorem*, φ corresponds to a vector $y \in \mathbb{R}^n$ such that $\forall x \in \mathbb{R}^n : \varphi(x) = \langle y, x\rangle$. This y stands orthogonal on the levels sets of φ, and points in the direction where φ increases. The longer y is, the narrower are the level sets, the shorter is the wavelength of the corresponding plane wave.



We can think of φ as a machine, that takes a vector $x \in \mathbb{R}^n$, computes the number of level sets that are pierced by x (where we consider only the level sets $0, 1, 2, \dots$), and outputs this number as $\varphi(x)$. In particular $\varphi(y) =$ (number of level sets pierced by y) $= \|y\|^2$, because the levels sets have the distance $\frac{1}{\|y\|}$, and y is orthogonal to the level sets. Note that this is in accordance to $\varphi(y) = \langle y, y\rangle = \|y\|^2$.



For whom who study physics: The duality "linear functional $\varphi \in (\mathbb{R}^3)^* \leftrightarrow$ vector $y \in \mathbb{R}^3$" is similar to the nature of light waves in physics. The levels sets of φ correspond to the wavefronts of the plane wave, and the vector y corresponds to the momentum vector of the wave (in appropriate units). //

2.7 Bases of Hilbert Spaces – Motivation

We have Hilbert space $\mathcal{H}$. We pick any $e_1 \in \mathcal{H}$ with $\|e_1\| = 1$, then pick $e_2 \in \{e_1\}^\perp$ with $\|e_2\| = 1$, and continue. We get a sequence $(e_1, e_2 \dots, e_n, \dots)$.

Remark: Index sets don't have to be countable, they can be any arbitrary set.

Remark: Hilbert spaces with countable many directions are called seperable, and otherwise not separable.

2.8 Digression: Zorn's Lemma

Definition 2.58 *(partial order, linear order, upper bound, maximal element)***.**

- A relation $x \preceq y$ on a set S is called *partial order*, if it is reflexive, transitive, and anti-symmetric (i.e. $x \preceq y \wedge y \preceq x \Rightarrow x = y$).
- A set S is *linearly ordered*, if for each $x, y \in S$ either $x \preceq y$ or $y \preceq x$.
- An element $p \in S$ is called an *upper bound* of a subset $O \subseteq S$, if for each $x \in O$ it holds that $x \preceq p$.
- An element $m \in S$ is called *maximal element*, if for each $x \in S$ it holds that $m \preceq x \Rightarrow m = x$.

Example 2.59 *(example for a partial order)***.**

(1) $S = 2^X$ and $A \preceq B :\Leftrightarrow A \subseteq B$. ◊

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Statement 2.60 *(Axiom of Choice)*. Function $g\colon A \to$ set of sets. *AC*: Suppose that $\forall x \in A : g(x) \neq \emptyset$. Then exists a f with $\forall x \in A : f(x) \in g(x)$. □

In Zeremlo-Fraenkl-set theory, equivalent to Axiom of Choice is Zorn's lemma:

Statement 2.61 *(Zorn's lemma)*. Let $(S, \leq)$ be a partial ordered set. Assume that each linearly ordered subset has an upper bound. Then each linearly ordererd subset has an upper bound that is a maximal element. □

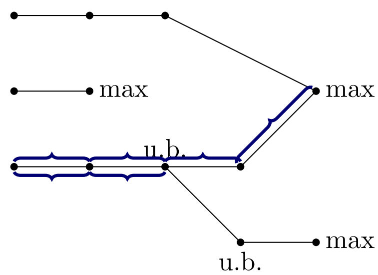

Figure 3: A partial ordered set S. Marked are two linearly ordered subsets O_1, O_2 (as blue braces), two upper bounds of O_1, all four maximal elements of S.

Example 2.62 *(applicability of Zorn's lemma for "$\subseteq$")*. $\Sigma \subseteq 2^X$, suppose that Σ is closed on taking unions. We order it, $(\Sigma, \leq)$, $A_1 \leq A_2 :\Leftrightarrow A_1 \subseteq A_2$. Then each linearly ordered subset $\{A_\alpha\}_\alpha$ has upper bound $\bigcup_\alpha A_\alpha$. ◊

2.9 Digression: Infinite Sums

Remark by the typesetter: This section was rewritten by the typesetter of the script, and hence does not correspond 1:1 to the lecture.

Definition 2.63 *(infinite sums – definitions)*.

- Sum of a sequence (real analysis):
  Let X be a normed space, and denote natural numbers by $\mathbb{N}$.
  Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be a sequence.
  Define the sum $\sum_{n\in\mathbb{N}} x_n$ as the limit of the sequence $(\sum_{n=1}^N x_n)_{N\in\mathbb{N}} \in X^{\mathbb{N}}$, i.e. $\sum_{n\in\mathbb{N}} x_i = x$ iff

$$\forall \varepsilon > 0 \quad \exists N_0 \in \mathbb{N} \quad \forall N \geq N_0 : \quad \left\| \sum_{n=1}^N x_n - x \right\| < \varepsilon.$$

  We say $\sum_{n\in\mathbb{N}} x_n$ is absolute convergent, iff $\sum_{n\in\mathbb{N}} |x_n|$ converges.

- Sum of a measureable function (measure theory):
  Let Ω be countable set, denote counting measure by μ, consider measure space $(\Omega, \mathcal{P}(\Omega), \mu)$.
  Let $(x_\omega)_{\omega\in\Omega} \in \mathbb{R}^\Omega$ be a measureable function.
  Define the sum $\sum_{\omega\in\Omega} x_\omega$ as the integral $\int_{\omega\in\Omega} x_\omega \ \mu(\mathrm{d}\omega)$, i.e. $\sum_{\omega\in\Omega} x_\omega = x$ exists iff

$$x = \underbrace{\int_\omega (x_+)_\omega \ \mu(\mathrm{d}\omega)}_{\text{always exists}} - \underbrace{\int_\omega (x_-)_\omega \ \mu(\mathrm{d}\omega)}_{\text{always exists}} \text{ determined.}$$

- Sum of a family (functional analysis):
  Let X be a Banach space, and I an arbitrary set.
  Let $(x_i)_{i\in I} \in X^I$ be a family.
  We say $\sum_{i\in I} x_i = x$ iff

$$\forall \varepsilon > 0 \quad \exists F_0 \subseteq I \text{ finite} \quad \forall F \supseteq F_0 \text{ finite} : \quad \left\| \sum_{i\in F} x_i - x \right\| < \varepsilon.$$

We say $\sum_{i\in I} x_i$ is absolute convergent, iff $\sum_{n\in I} |x_i|$ converges.

Lemma 2.64 *(infinite sums – equivalance of the definitions)***.** In the notation of definition 2.63 (denote $(I) := \{i \in I \mid x_i \neq 0\}$):

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} x_n \begin{array}{l}\text{convergent, but not}\\ \text{absolute convergent}\end{array} &&\Rightarrow && \forall x\in X\ \exists J\colon \mathbb{N}\to\mathbb{N} \text{ bijection}: \sum_{n\in\mathbb{N}} x_{J(n)} = x \qquad \begin{bmatrix}\text{only for}\\ X=\mathbb{R}!\end{bmatrix}\\
&\sum_{n\in\mathbb{N}} x_n \text{ absolute convergent} &&\Rightarrow && \forall J\colon \mathbb{N}\to\mathbb{N} \text{ bijection}: \sum_{n\in\mathbb{N}} x_{J(n)} = \sum_{n\in\mathbb{N}} x_n\\
&\sum_{\omega\in\Omega} x_\omega \text{ determined} &&\Leftrightarrow && \exists J\colon \mathbb{N}\to\Omega \text{ bijection}: \sum_{n\in\mathbb{N}} x_{J(n)} \begin{array}{l}\text{absolute}\\ \text{convergent}\end{array}\\
&\sum_{x\in I} x_i \text{ absolute convergent} &&\Leftrightarrow && \exists J\colon \mathbb{N}\to (I) \text{ bijection}: \sum_{n\in\mathbb{N}} x_{J(n)} \begin{array}{l}\text{absolute}\\ \text{convergent}\end{array}
\end{aligned}$$

Note that the latter "$\exists J\colon \mathbb{N}\to(I)$ bijection" says that, in this case, at most countable x_i's are nonzero. □

Prop. 2.65 *(properties of the "functional analysis definition")***.**

(a) If $\forall i\in I: x_i\geq 0$, then $\sum_{i\in I} x_i$ converges if and only if $\sup_{F\subseteq I \text{ finite}} \sum_{i\in F} x_i < \infty$.

(b) If $\forall i\in I: x_i \geq 0$ and $\sum_{i\in I} x_i$ converges, then only countable many x_i's are nonzero. □

Proof. Proof of (b): Let $I_n = \{i\in I \mid x_i > \frac{1}{n}\}$. Then $\bigcup_{n\in\mathbb{N}} I_n = \{i\in I\mid x_i > 0\}$. If the righthand side is uncountable, then there exists a N such that I_N is infinite. Then clearly $\sup_{F\subseteq I_n} \sum_{i\in F} x_i = \infty$. ■

2.10 Bases of Hilbert Spaces

Definition 2.66 *(orthonormal basis)***.** An orthonormal set $S = \{e_\alpha\}_{\alpha\in A}, e_\alpha \in \mathcal{H}$, then S is called an *orthonormal basis*, if any orthonormal set $S' \subseteq S$ implies $S' = S$.

Remark 2.67. An orthonormal basis don't have to be a (linear algebra) basis of $\mathcal{H}$. //

Theorem 2.68 *(every Hilbert space has an orthonormal basis)***.** Every Hilbert space has an orthonormal basis. □

Proof. Let S_1, S_2 be two orthonormal sets. We order them by inlusion, $S_1 \leq S_2$ if $S_1\subseteq S_2$. (Set of all orthonormal sets, $\leq$) is a partially ordered set. Each linearly ordered chain $\{S_\alpha\}_{\alpha\in I}$ then $\bigcup_{\alpha\in I} S_\alpha$ is an upper bound. It follows with Zorn's lemma that there exists a maximal orthonormal set S. Being maximal means that if $S'\subseteq S$ then $S' = S$. ■

Theorem 2.69 *(properties of orthonormal basis)***.** Let $S = \{e_\alpha\}_{\alpha\in A}$ be an orthonormal basis. Then the following holds:

(1) Coordinate representation: Every vector $x\in\mathcal{H}$ can be represented as

$$x = \sum_{\alpha\in A} e_\alpha \langle e_\alpha, x\rangle.$$

(2) Parseval identity: For every vector $x \in \mathcal{H}$, the so called *Parseval identity* holds,

$$\|x\|^2 = \sum_{\alpha\in A} |\langle e_\alpha, x\rangle|^2.$$

(3) Let $(c_\alpha)_{\alpha\in A} \in \mathbb{F}^A$ be an arbitrary family. Then (both sums in the "functional analysis"-sense)

$$\underbrace{\sum_{\alpha\in A} {c_\alpha}^2 < \text{converges}}_{\text{"coordinates" converges absolutely}} \quad\Rightarrow\quad \underbrace{\sum_{\alpha\in A} c_\alpha e_\alpha \text{ converges}}_{\text{"infinite linear combination" converges}} \quad .$$
□

Proof. Let $F\subseteq A$ be a finite set, then by Bessel inequality, $\sum_{\alpha\in F} |\langle e_\alpha, x\rangle|^2 \leq \|x\|^2$, and therefore

$$\sum_{\alpha\in A} |\langle e_\alpha, x\rangle|^2 \leq \|x\|^2 \text{ converges.}$$

By virtue of (b) above, it follows that $\langle e_\alpha, x\rangle \neq 0$ only for countable many elements, $\alpha_1, \alpha_2, \alpha_3, \dots$. We have $\sum_{j\in\mathbb{N}} |\langle e_\alpha, x\rangle|^2 \leq \|x\|^2$. We claim $x_n := \sum_{j=1}^n e_{\alpha_j}\langle e_{\alpha_j}, x\rangle$ is Cauchy sequence. Let $n \geq m$. Then

$$\|x_n - x_m\|^2 = \left\| \sum_{j=m}^{n} e_{\alpha_j}\langle e_{\alpha_j}, x\rangle \right\|^2 = \sum_{j=n}^{m} \left|\langle e_{\alpha_j}, x\rangle\right|^2,$$

and hence $(x_n)_n$ is a Cauchy sequence. Because $\mathcal{H}$ is a Banach space, it follows that $x_n \longrightarrow \tilde{x}$.

$$\begin{aligned}\langle e_{\alpha_j}, x - \tilde{x}\rangle &= \lim_{N\to\infty} \langle e_{\alpha_j}, x - x_N\rangle \\ &= \lim_{N\to\infty} \left\langle e_{\alpha_j}, x - \sum_{k=1}^{N} e_{\alpha_k}\langle e_{\alpha_k}, x\rangle \right\rangle \\ &= \langle e_{\alpha_j}, x\rangle - \langle e_{\alpha_j}, x\rangle \\ &= 0\end{aligned}$$

If $\alpha \neq \alpha_j$, then also $\langle e_\alpha, x - \tilde{x}\rangle = 0$. Then for all $\alpha \in A, e_\alpha \in S$, $\langle e_\alpha, x - \tilde{x}\rangle = 0$. Therefore $x - \tilde{x} = 0$, because otherwise $S \cup \{\frac{x-\tilde{x}}{\|x-\tilde{x}\|}\}$ is an orthonormal set.

$$\begin{aligned}\left\| x - \sum_{j=1}^{N} \langle e_{\alpha_j}, x\rangle e_{\alpha_j} \right\|^2 &= \left\langle x - \sum_{j=1}^{N} \langle e_{\alpha_j}, x\rangle e_{\alpha_j}, x - \sum_{k=1}^{N} \langle e_{\alpha_k}, x\rangle e_{\alpha_k} \right\rangle \\ &= \|x\|^2 - 2\sum_{k=1}^{N} \left|\langle e_{\alpha_k}, x\rangle\right|^2 + \sum_{k=1}^{N} \left|\langle e_{\alpha_k}, x\rangle\right|^2 \\ &= \|x\|^2 - \sum_{k=1}^{N} \left|\langle e_{\alpha_k}, x\rangle\right|^2\end{aligned}$$

$$\begin{aligned}0 &= \lim_{N\to\infty} \left\| x - \sum_{k=1}^{N} e_{\alpha_k}\langle e_{\alpha_k}, x\rangle \right\|^2 \\ &= \lim_{N\to\infty} \left(\|x\|^2 - \sum_{k=1}^{N} \left| e_{\alpha_k}\langle e_{\alpha_k}, x\rangle\right|^2 \right)\end{aligned} \qquad \therefore \quad \|x\|^2 = \sum_{k=1}^{\infty} \left|\langle e_{\alpha_k}, x\rangle\right|^2$$

Steps:

1. Only countable many c_α is non-zero
2. Prove that partial sums $\sum_{j=1}^N c_{\alpha_j} e_{\alpha_j}$ is Cauchy
3. If Cauchy, then convergent. ■

Recap:

Theorem 2.70 *(characterization of orthonormal basis)***.** Let $S = \{e_\alpha\}_{\alpha\in A}$ be an orthonormal set. Then each of the following statements is equivalent to "S is a basis":

(i) $\forall S'$ orthonormal set : $S' \supseteq S \;\Rightarrow\; S' = S$

(ii) $S^\perp = \{0\}$, i.e. $\forall x \in \mathcal{H} :\; (\forall \alpha \in A : \langle x, e_\alpha\rangle = 0) \;\Rightarrow\; x = 0$

(iii) $\overline{\operatorname{span} S} = \mathcal{H}$

(iv) $\forall x \in \mathcal{H} :\; \|x\|^2 = \sum_{\alpha\in A} |\langle x, e_\alpha\rangle|^2$

(v) $\forall x \in \mathcal{H} :\; x = \sum_{\alpha\in A} e_\alpha \langle e_\alpha, x\rangle$

(vi) $\forall x, y \in \mathcal{H} :\; \langle x, y\rangle = \sum_{\alpha\in A} \overline{\langle e_\alpha, x\rangle} \cdot \langle e_\alpha, y\rangle$ □

Proof. We proved the hard parts in the last lecture.

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"(v) ⇒ (vi)":

$$\begin{aligned}\langle x,y\rangle &= \left\langle \sum_\alpha e_\alpha\langle e_\alpha,x\rangle, \sum_\beta e_\beta\langle e_\beta,y\rangle\right\rangle\\ &= \sum_{\alpha,\beta}\overline{\langle e_\alpha,x\rangle}\cdot\langle e_\beta,y\rangle\cdot\langle e_\alpha,e_\beta\rangle\\ &= \sum_\alpha\overline{\langle e_\alpha,x\rangle}\cdot\langle e_\alpha,y\rangle\end{aligned}$$

$$\lim_{N\to\infty}\left\langle\sum_{j=1}^N e_{\alpha_j}\langle e_{\alpha_j},x\rangle, \sum_{k=1}^N e_{\beta_k}\langle e_{\beta_k},x\rangle\right\rangle = \left\langle\sum_{j=1}^\infty e_{\alpha_j}\langle e_{\alpha_j},x\rangle, \sum_{k=1}^\infty e_{\beta_k}\langle e_{\beta_k},x\rangle\right\rangle$$

■

Definition 2.71 *(separable space)***.** A topological space X is called *separable*, if it contains a countable dense subset S,

$$S = \{x_n\}_{n=1}^\infty \in X^{\mathbb{N}} \quad \text{and} \quad \overline{S} = X.$$

Algorithm 2.72 *(Gram-Schmidt orthonormalization)***.** Let $\{v_n\}_{n=1}^\infty$ be a set of independent vectors. Define recursively:

$$\begin{aligned} w_1 &= v_1, & e_1 &= \frac{w_1}{\|w_1\|}\\ w_{n+1} &= v_{n+1} - \sum_{j=1}^n e_j\langle e_j, v_{n+1}\rangle, & e_{n+1} &= \frac{w_{n+1}}{\|w_{n+1}\|}\end{aligned}$$

Then:

(1) $\{e_j\}_{j=1}^N$ is orthonormal

(2) $\operatorname{span}\{v_j\}_{j=1}^n = \operatorname{span}\{e_j\}_{j=1}^n$ for any $1 \le n \le N$ □

Theorem 2.73 *(characterization of separable Hilbert spaces)***.** A Hilbert space is separable iff it has countable orthogonal basis. □

Proof. Proof of "⇒": $\overline{\{x_n\}_{n=1}^\infty} = \mathcal{H}$

1. Get sequence $\{v_n\}_{n=1}^N$ (where $N \in \mathbb{N}_0 \cup \{\infty\}$) of linearly independent vectors such that $\overline{\{v_n\}_{n=1}^N} = \mathcal{H}$
2. Now do Gram-Schmidt orthgonalization process to get $S = \{e_n\}_{n=1}^\infty$, by construction $\overline{\operatorname{span}(S)} = \mathcal{H}$

Proof of "⇐": Consider alls rational finite linear combinations of basis vectors (see exercise). ■

Corollary 2.74 *(coordinate representation is isometry)***.** A separable infinite-dimensional Hilbert space $\mathcal{H}$ is isometric to ℓ^2. A finite-dimensional Hilbert space is isometric to $\mathbb{C}^n$ for some n. □

Proof. Separable Hilbert space has a basis $\{e_n\}_{n=1}^\infty$. Define map

$$\mathcal{H} \to \ell^2,\ x \mapsto \{\langle e_n, x\rangle\}_{n=1}^\infty,$$

then:

- Well-defined because of Bessel inequality
- Isometry because of Parseval identity ($\|x\|_{\mathcal{H}} = \|\{\langle e_n,x\rangle\}_{n=1}^\infty\|_{\ell^2}$)
- Bijective because of ... ■

2.11 [Digression] Applications

2.11.1 Measure theory

Theorem 2.75 *(Radon-Nikodym)***.** Let μ, ν be finite measures on a measureable space (X, Σ). Suppose that ν is absolutely continuous w.r.t. μ, then there exists a g μ-measureable and $g \geq 0$ such that

$$\forall E \in \Sigma : \ \nu(E) = \int_E g \, \mathrm{d}\mu \,,$$

what is equivalent to

$$\int_X f \, \mathrm{d}\nu = \int_X (f \cdot g) \mathrm{d}\mu.$$

g is called the *Radon-Nikodym derivative*, "$\mathrm{d}\nu = g \ \mathrm{d}\mu$". □

Remark 2.76. The theorem also holds for σ-finite measures. Recall:
- Finite: $\mu(X), \nu(X) < \infty$
- σ-finite: ...
- Absolutely continuous $\nu \ll \mu$: $\forall F \in \Sigma : \ \mu(F) = 0 \ \Rightarrow \ \nu(F) = 0$ //

Proof by von Neumann. $L^2(X, \mu + \nu)$ is a (real) Hilbert space,

$$\langle f, g \rangle = \int_X (f \cdot g) \ (\mathrm{d}\nu + \mathrm{d}\mu), \quad \|f\| = \sqrt{\int_X f^2 \ (\mathrm{d}\nu + \mathrm{d}\mu)}.$$

Consider a functional $f \mapsto \int_X f \ \mathrm{d}\mu$. Claim: This is a bounded functional $\mathcal{H} \to \mathbb{R}$.

$$\left| \int_X f \ \mathrm{d}\mu \right| \leq \sqrt{\int_X f^2 \ \mathrm{d}\mu} \cdot \sqrt{\int_X \mathrm{d}\mu} \leq \sqrt{\int_X f^2 \ (\mathrm{d}\mu + \mathrm{d}\nu) \cdot \mu(X)}$$

By virute of the Riesz representation theorem ("$\mathcal{H}^\star = \mathcal{H}$"), there exists a function h such that

$$\int_X f \ \mathrm{d}\mu = \int_X (fg) \ (\mathrm{d}\mu + \mathrm{d}\nu)$$
$$\int_x f(1-h) \ \mathrm{d}\mu = \int_X (fh) \ \mathrm{d}\nu. \qquad (*)$$

Define function $\tilde{f}$ such that $f = \tilde{f}\frac{1}{h}$. Claim $0 < h \leq 1$ almost surely:

- Let $F := \{x \mid h(x) \leq 0\}$. Put $f =$ characteristic function of F into $(*)$:

$$\mu(F) \leq \int_F (1-h) \ \mathrm{d}\mu = \int_F h \ \mathrm{d}\nu \leq 0 \quad \therefore \quad \mu(F) \leq 0 \quad \therefore \quad \mu(F) = 0 \quad \therefore \quad \nu(F) = 0 \quad \therefore \quad (\mu + \nu)(F) = 0$$

- Let $F = \{x \mid h(x) > 1\}$. Put characterisitic function of F into $(*)$,

$$\int_F (1-h) \ \mathrm{d}\mu = \int_F h \mathrm{d}\nu.$$

  Suppose that $\mu(F) > 0$, then the left hand side is negative, but the right hand side is non-negative. Contradiction, hence $\mu(F) = 0$, and therefore $(\mu + \nu)(F) = 0$.

Put $f = \tilde{f}\frac{1}{h}$ into $(*)$,

$$\int_X \tilde{f}\frac{1-h}{h} \ \mathrm{d}\mu = \int_X \tilde{f} \ \mathrm{d}\nu.$$

Conclusion: $g = \frac{1-h}{h}$ satisfies the theorem. ■

2.11.2 Fourier transform

Classical result in *Fourier theory*:

Definition 2.77 *(Fourier coefficients)***.** To each function f, define the fourier coefficients of f to be

$$c_n := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} \mathrm{e}^{\mathrm{i}nx} f(x) \ \mathrm{d}x, \quad n \in \mathbb{Z}.$$

Theorem 2.78 *(Fourier series – classical viewpoint).* For every 2π-periodic function $f \in C(]-\pi, +\pi[)$, its Fourier series converges uniformly to f,

$$\frac{1}{\sqrt{2\pi}} \sum_{j=-N}^{+N} c_j \mathrm{e}^{\mathrm{i}jx} \xrightarrow[\text{uniformly}]{N\to\infty} f(x).$$

□

Theorem 2.79 *(Fourier series – functional analysis viewpoint).* Consider the space $L^2(]-1,+1[)$. Then $(e_n)_{n\in\mathbb{N}}$, $e_n(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{inx}$ is an orthonormal basis of $L^2(]-1,+1[)$, i.p. $\forall m \neq n : \langle e_m, e_n\rangle = 0$ and $\forall n : \langle e_n, e_n\rangle = 1$. Therefore, for every function $f \in L^2(]-1,+1[)$

$$\sum_n c_n e_n \xrightarrow[L^2\text{-conv.}]{} f \quad \text{where} \quad c_n := \langle e_n, f\rangle \quad \text{i.e.} \quad f \underset{L^2\text{-eq.}}{=\!=\!=} \sum_{n=-\infty}^{n=+\infty} e_n \langle e_n, f\rangle,$$

where the latter equality is in the L^2-sense, not pointwise equality. □

Proof. Use Stone-Weierstrass theorem, to get that $S = \{e_n\}_{n=1}^{\infty}$ is dense in $C(]-\pi,+\pi[)$. ■

Bounded Operators 3

2015-05-19

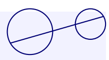
3.1 Bounded Linear Maps

M, N normed linear spaces (over the same field $\mathbb{F}$).

Definition 3.1 *(continuity, linearity, boundedness of maps)*. Let $L\colon M \to N$ be a map.

- L is called *linear*, if $\forall \alpha \in \mathbb{F},\ x, y \in M:\ L(x + \alpha y) = L(x) + \alpha L(y)$
- L is called *sequential continuous*, if $x_n \xrightarrow[\text{in } M]{n\to\infty} x \ \Rightarrow\ L(x_n) \xrightarrow[\text{in } N]{n\to\infty} L(x)$.
  Note that in metric spaces, *continuity* is equivalent to sequential continuity.
- L is called *bounded*, if $\exists C > 0:\ \|L(x)\|_N \le C \cdot \|x\|_M$.
  This condition is equivalent to $\sup_{\|x\|_M = 1} \|L(x)\|_N < \infty$.

Definition 3.2 *(diameter, boundedness of sets)*. Set S is *bounded* if $\operatorname{diam}(S) := \sup_{x,y \in S} \|x - y\|_M < \infty$.



Prop. 3.3 *(characterization of bounded maps)*. A map L is bounded iff it maps bounded sets to bounded sets. □

Proof. Proof of "⇒":

$$\operatorname{diam}(L[S]) = \sup_{x,y \in S} \|L(x) - L(y)\|_N \le C \sup_{x,y \in S} \|x - y\|_M = C \operatorname{diam}(S)$$

Proof of "⇐": $L[B_1]$ is bounded set then $\operatorname{diam}(L[B_1]) < \infty$:

$$\sup_{\|x\|_M = 1} \|L(x)\|_N \le \operatorname{diam}(L[B_1]) < \infty$$

■

Theorem 3.4 *(characterization of continuity for linear maps)*. Let L be a linear map $M \to N$. Then the following is equivalent:

(i) L is continuous

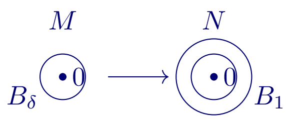
(ii) L is continuous at 0

(iii) L is bounded □

Proof.

- "(i) ⇒ (ii)": clear.
- "(ii) ⇒ (iii)": Because f is continuous at 0, there exists a $\delta > 0$ such that $\|x\|_M \le \delta \ \Rightarrow\ \|L(x)\|_N \le 1$. Then

$$\sup_{\|x\|_M = 1} \|L(x)\|_N = \tfrac{1}{\delta} \sup_{\|x\|_M = 1} \|L(\delta x)\|_N \le \tfrac{1}{\delta} < \infty.$$



- "(iii) ⇒ (i)": Because f is bounded, there exists a C such that .... Pick $\|x - y\|_M \le \frac{\varepsilon}{C} = \delta$, then

$$\|L(x - y)\|_N \le C \|x - y\|_M = \varepsilon.$$

■

Definition 3.5 *(space of all bounded linear maps, operator norm)*. Let $\mathcal{L}(M, N)$ denote the space of all bounded linear maps from M to N. The elements of $\mathcal{L}(M, N)$ are called *bounded operators*. For the special case $M = N$ we also write $\mathcal{L}(M, N) = \mathcal{B}(M)$. We equip $\mathcal{L}(M, N)$ with the so-called *operator norm* $\|\cdot\|_{M \to N}$,

$$\|\cdot\|_{M \to N} := \sup_{\|x\|_M = 1} \|Lx\|_N < \infty.$$

Definition 3.6 *(dual space)***.** Recall definition 3.5 and consider the special case $N = \mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Then $\mathcal{L}(M,\mathbb{F}) = M^*$ is the *dual space* of M, and the elements of $\mathcal{L}(M,\mathbb{F})$ are the *linear functionals* on M.

Recall:
Definition 2.48 *(dual space)***.** A map $\varphi\colon \mathcal{H} \to \mathbb{C}$ is called a *linear functional*, if it is a bounded linear map, i.e.:
(1) Linearity: $\forall x, y \in \mathcal{H},\ \alpha \in \mathbb{C}:\ \varphi(x + \alpha y) = \varphi(x) + \alpha\varphi(y)$
(2) Boundedness: $\exists C \in \mathbb{R}:\ |\varphi(x)| \le C\|x\|_{\mathcal{H}}$
The space of all linear functionals on $\mathcal{H}$ is called the *dual space* $\mathcal{H}^*$ of $\mathcal{H}$. We equip $\mathcal{H}^*$ with a norm $\|\cdot\|_{\mathcal{H}^*}$,

$$\|\varphi\|_{\mathcal{H}^*} := \sup_{x\in\mathcal{H},\|x\|=1} |\varphi(x)| = \sup_{x\in\mathcal{H},x\neq 0} \frac{|\varphi(x)|}{\|x\|}.$$

Remark 2.49. *Remark by the typesetter: This definition holds for any normed space, not just Hilbert spaces. Anyway, the more general definition will come in definition 3.6. Furhtermore, the norm $\|\cdot\|_{M^*}$ conincides with the operator norm $\|\cdot\|_{M\to\mathbb{F}}$.* //

Notation 3.7. Sometimes, we omit braces "(", ")" and the composition symbol "$\circ$":

- For $L\colon M \to N$ linear map and $x \in M$, we write $L(x) := Lx$.
- For $L_1\colon M_1 \to M_2$ and $L_2\colon M_2 \to M_3$, we write $L_2L_1 := L_2 \circ L_1\colon M_1 \to M_3$. //

Two inequalities about $\|\cdot\|_{M\to N}$:
Theorem 3.8 *(submultiplicativity of the operator norm)***.**

(1) $\|Lx\|_N \le \|L\|_{M\to N}\|x\|_M$.

(2) $\|L_2L_1\|_{M_1\to M_3} \le \|L_2\|_{M_2\to M_3}\|L_1\|_{M_1\to M_2}$ □

Proof.

(1) $$\|Lx\|_N \le \sup_{\|y\|_M=1} L\left(y\|x\|_M\right) = \|x\|_M\|L\|_{M\to N}$$

(2) $$\|L_2L_1\|_{M_1\to M_3} = \sup_{\|x\|_{M_1}=1} \|L_2L_1x\|_{M_3} \le \sup_{\|x\|_{M_1}=1} \|L_2\|_{M_2\to M_3}\|L_1x\|_{M_2} = \|L_2\|_{M_2\to M_3}\|L_1\|_{M_1\to M_2}$$ ■

Theorem 3.9 *(properties of $\mathcal{L}(M,N)$)***.** The space $(\mathcal{L}(M,N), \|\cdot\|_{M\to N})$ is a normed linear space. And if N is a Banach space, then so is $\mathcal{L}(M,N)$. □

Proof. $\|\cdot\|_{M\to N}$ is a norm:

$$\|L_1 + L_2\|_{M\to N} = \sup_{\|x\|_M=1} \|(L_1+L_2)x\|_N \le \sup_{\|x\|_M=1} \|L_1x\|_N + \sup_{\|x\|_M=1} \|L_2x\|_N = \|L_1\|_{M\to N} + \|L_2\|_{M\to N}$$

Consider Cauchy sequence $(L_n)_{n=1}^\infty$,

$$\|L_n - L_k\|_{M\to N} \le \varepsilon \text{ if } n, k \text{ is large.}$$

Then for each $x \in M$, $(L_nx)_n$ is Cauchy sequence in N,

$$\|L_nx - L_kx\|_N \le \|L_n - L_k\|_{M\to N}\|x\|_M \le \varepsilon\|x\|_M.$$

Because N is a Banach space, it follows that $Lx := \lim_{n\to\infty} L_nx$ exists for each $x \in M$.

- Linearity: $L(x+y) = \lim_{n\to\infty} L_n(x+y) = \lim_{n\to\infty} L_nx + L_ny = Lx + Ly$
- Boundedness: Observe $(\|L_n\|_{M\to N})_n$ is a Cauchy sequence, $|\|L\| - \|\tilde{L}\|| \le \|L - \tilde{L}\|$. If $(\|L_n\|_{M\to N})_n$ is Cauchy, then there is a $C > 0$ such that $\forall n \in \mathbb{N}: \|L_n\|_{M\to N} \le C$. Then we have $\sup_{\|x\|_M=1}\|Lx\|_N = \sup_{\|x\|_M=1} \lim_{n\to\infty}\|L_nx\|_N \le \sup_{\|x\|_M=1} \lim_{n\to\infty} C\|x\|_M = C < \infty$.

Let n be such that for all $k \ge n$ it holds that

$$\begin{aligned} &\forall x \in M: \lim_{k\to\infty} \|(L_n - L_k)x\|_N \le \varepsilon\|x\|_M \\ \therefore\quad &\|(L_n - L)x\|_N \le \varepsilon\|x\|_M \\ \therefore\quad &\sup_{\|x\|_M=1} \|(L_n - L)x\|_N \le \varepsilon \\ \therefore\quad &\|L_n - L\|_{M\to N} \le \varepsilon \end{aligned}$$ ■

Example 3.10 *(examples of linear maps)***.**

(1) Consider $M = C([-1,+1])$ and a linear functional $\varphi \in M^*$ defined by $\varphi(f) = f(0)$. Then $|\varphi(f)| \le \|f\|_M$, and hence $\|\varphi\|_{M^\star} \le 1$, and actually $\|\varphi\|_{M^\star} = 1$.

(2) Consider $M = C([0,+1])$ and continuous function $K\colon [0,+1]\times[0,+1] \to \mathbb{C}$, then $(Lf)(x) := \int_0^1 K(x,y) f(y)\,\mathrm{d}y$ is an operator in $\mathcal{L}(M)$.

$$\|Lf\|_M = \sup_{x\in[0,1]} |(Lf)(x)| = \sup_{x\in[0,1]} \left| \int_0^1 K(x,y) f(y)\,\mathrm{d}y \right| \le \sup_{x,y\in[0,1]} |K(x,y)| \|f\|_M \quad \therefore \quad \|L\|_{M\to M} \le \sup_{x,y\in[0,1]} |K(x,y)|$$

◊

2015-05-22

Question: Let $L\colon M \to N$ be a bounded norm, $L \in \mathcal{L}(M,N)$, and consider the norm $\|\cdot\|_{M\to N}$. Is $\|L\|_{M\to N} = \sup_{x\in M, \|x\|_M \le 1} \|Lx\|_N$ a correct relation?

3.2 Digression: Unbounded operators

2015-05-19

Remark 3.11 *(unbounded maps)***.**

- unbounded $\ne$ not bounded
- unbounded = not defined everywhere (very important)
- discontinuous = not bounded (obscurity) //

Definition 3.12 *(Hamel basis)***.** *Hamel basis* (*algebraic basis*) of M: This is a set $S = \{e_\alpha\}_{\alpha\in A}$ satisfying:

- Any finite subset of S is linearly independent
- All $x \in M$ can be *uniquely* written as *finite* linear combination of $\{e_\alpha\}_{\alpha\in A}$

Prop. 3.13 *(every linear space has an algebraic basis)***.** Every normed linear space M has an algebraic basis. □

Remark 3.14. If M is a Banach space and $\dim M = \infty$, then the Hamel basis is uncountable. //

Prop. 3.15 *(existence of discontinuous maps)***.** Not bounded maps do exist. □

Proof. Let M be a Banach space of $\dim M = \infty$. Pick a countable sequence $(e_{\alpha_n})_{n=1}^\infty$ (w.l.o.g. $\|e_{\alpha_n}\| = 1$). Define $L\colon M \to \mathbb{C}$ by $Le_{\alpha_n} = n$, and $Le_\alpha = 0$ if $e_\alpha \ne e_{\alpha_n}$ for any n, and linearity. Then L is linear, but clearly not bounded. ■

3.3 The Dual Space of a ℓ^p-Space

2015-05-22

Consider ℓ^p, at first only $p \in \,]1,\infty[$, and $p \in \{1,\infty\}$ later.

Theorem 3.16 *(Hölder inequality)***.** For $x \in \ell^p$ and $y \in \ell^q$, where p, q conjugate numbers, e.g. $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left|\sum_{n=1}^\infty x_n y_n\right| \le \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p} \cdot \left(\sum_{n=1}^\infty |x_n|^q\right)^{1/q} = \|x\|_p \cdot \|y\|_q.$$ □

Proof. Omitted. ■

Lemma 3.17 *(every vector in ℓ^q induces a linear functional in $(\ell^p)^*$)***.** For $y \in \ell^q$, define

$$\varphi\colon \ell^p \to \mathbb{C}, \; \varphi_y(x) := \sum_{n=1}^\infty x_n y_n.$$

Then $\varphi_y \in (\ell^p)^*$, i.e. φ_y is bounded. □

Proof.

$$\|\varphi_y\| = \sup_{\|x\|_p=1} |\varphi_y(x)| = \sup_{\|x\|_p=1} \left|\sum_{n=1}^{\infty} x_n y_n\right| \leq \sup_{\|x\|_p=1} \|x\|_p \|y\|_q = \|y\|_q$$ ∎

Lemma 3.18 *(norm of induced functional)***.** For every $y \in \ell^q$, it holds that

$$\|\varphi_y\|_{(\ell^p)^*} = \|y\|_{\ell^q}.$$ □

Proof. From the proof of lemma 3.17 we know $\|\varphi_y\|_{(\ell^p)^*} \leq \|y\|_{\ell^q}$. Furthermore, for any $\|z\|_p = 1$, $\|\varphi_y\| = \sup_{\|x\|_p=1}|\varphi_y(x)| \geq |\varphi_y(z)|$. We claim that equality is achieved if $|x_n|^p = |y_n|^q$, i.e. $|x_n| = |y_n|^{q/p}$. Proof of claim: Take $\tilde{z} = |y_n|^{q/p}\operatorname{sgn}(y_n)$, then $\tilde{z} \in \ell^p$, because $\|\tilde{z}\|_p{}^p = \sum_{n=1}^{\infty}|y_n|^q = \|y\|_q{}^q$. Take $z = \frac{\tilde{z}}{\|y\|_q{}^{q/p}}$, then

$$\varphi_y(z) = \sum_{n=1}^{\infty} \frac{|y_n|^{q/p}}{\|y\|_q{}^{q/p}} = \|y\|_q{}^{-q/p} \sum_{n=1}^{\infty} |y_n|^{q/p+1} = \|y\|_q{}^{-q/p}\|y\|_q{}^q = \|y\|_q.$$

We conclude $\|\varphi_y\|_{(\ell^p)^*} = \|y\|_{\ell^q}$. ∎

Lemma 3.19 *(duality between p- and q-norm)***.**

$$\|x\|_p = \sup_{\|y\|_q=1} \left|\sum_{n=1}^{\infty} x_n y_n\right| = \sup_{\|y\|_q=1} |\varphi_y(x)|$$ □

Proof. Righthand side is

$$\sup_{\|y\|_q=1} |\varphi_y(x)| \leq \sup_{\|y\|_q=1} \|\varphi_y\| \|x\|_p = \|x\|_p.$$

Pick $y_n = |x_n|^{p/q}\operatorname{sgn}(x_n)$, then $\|x\|_p = \sup_{\|y\|_q=1}|\varphi_y(x)|$. ∎

Lemma 3.19 can be used in convex optimization. Another application of lemma 3.19 is proving that the p-norm $\|\cdot\|_p$ is indeed a norm.

Corollary 3.20 *(Minkowsi inequality = triangle inequality for $\|\cdot\|_p$)***.** $\|\cdot\|_p$ satisfies the triangle inequality. □

Proof.

$$\|x_1 + x_2\|_p = \sup_{\|y\|_q=1} |\varphi_y(x_1+x_2)| \leq \sup_{\|y\|_q=1} \big(|\varphi_y(x_1)| + |\varphi_y(x_2)|\big) = \|x_1\|_p + \|x_2\|_p$$ ∎

Corollary 3.21 *(p-norm is a norm)***.** From corollary 3.20 it follows that $\|\cdot\|_p$ is a norm. □

Lemma 3.22 *(every linear functional in $(\ell^p)^*$ is induced by a vector in ℓ^q)***.** For all $\varphi \in (\ell^p)^*$, there exists a $y \in \ell^q$ such that $\forall x \in \ell^p : \varphi(x) = \varphi_y(x)$. □

Proof. Let $\varphi \in (\ell^p)^*$ and $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, etc.. Define y by $y_n := \varphi(e_n)$. Things to check:

1. $y \in \ell^q$:

$$\|y\|_q = \sup_{\|x\|_p=1} \left|\sum x_n y_n\right| = \sup_{\|x\|_p=1} \left|\sum_{n=1}^{\infty} x_n \varphi(e_n)\right| = \sup_{\|x\|_p=1} |\varphi(x)| \leq \|\varphi\| < \infty$$

2. $\varphi = \varphi_y$:
   By construction $\varphi = \varphi_y$ on $c_{\text{cpt}} \subseteq \ell^p$. We know that c_{cpt} is dense in $\ell^p, p < \infty$, so it follows that $\varphi = \varphi_y$ (if continuous map coincide on a dense subset, then they are the same everywhere). ∎

Corollary 3.23 *($(\ell^p)^*$ is isomorphic to ℓ^q)***.** $(\ell^p)^*$ is isomorphic to ℓ^q: By lemma 3.17 and lemma 3.22 every vector $y \in \ell^q$ corresponds to a linear functional $\varphi \in (\ell^p)^*$ (via $y \mapsto \varphi_y$), and vice versa. Furthermore, by lemma 3.18 this bijection $(y \mapsto \varphi_y)$ is isometric. □

Remark 3.24.

$$\begin{aligned} \|\varphi_y\|_{(\ell^p)^*} &= \sup_{x\in\ell^p, \|x\|_{\ell^p}=1} |\varphi_y(x)| && \text{by definition} \\ \|x\|_{\ell^p} &= \sup_{\varphi\in(\ell^p)^*, \|\varphi\|_{(\ell^p)^*}=1} |\varphi(x)| && \text{by claim} \end{aligned}$$

//

Remark 3.25. ("$\cong$" means isometric)

- $(\ell^1)^* \cong \ell^\infty$
- $(\ell^\infty)^*$ is more complicated, since c_{cpt} is *not* dense in ℓ^∞
- $(L^p(X, \Sigma, \mu))^* \cong L^q(X, \sum, \mu)$ for $p \in]1, \infty[$
- $(L^1(X, \Sigma, \mu))^* \cong L^\infty(X, \sum, \mu)$ if μ is σ-finite
- $(L^\infty(X, \Sigma, \mu))^* \cong \text{bq}(X, \sum) =$ space of all σ-finite bounded measures $\nu << \mu$
  Example: $(L^\infty([-1, +1])^*$ constains inter alia of:
  - For any $g \in L^1([-1, +1])$, $f \mapsto \int_{-1}^{+1} \int_{-1}^{+1} f(x) \cdot g(x) \, \mathrm{d}x$
  - Measures: "δ-function: $f \mapsto f(0)$"

//

3.4 Hahn-Banach Theorem

Prop. 3.26. Let M be a normed linear space and $x \in M$.

$$\|x\| = \sup_{\varphi \in M^*, \|\varphi\|=1} |\varphi(x)|$$

□

Proof of proposition 3.26 – Part 1/2. Steps:

1. $\sup_{\|\varphi\|=1} |\varphi(x)| \leq \sup_{\|\varphi\|=1} \|\varphi\| \|x\| = \|x\|$
2. Try to find $\|\varphi\| = 1$ such that $\varphi(x) = \|x\|$.

This is a constrait on $Y = \{\lambda x \mid \lambda \in \mathbb{F}\}$. We finish the proof later. ■

Theorem 3.27 *(Hahn-Banach theorem – real version)*. Let X be a linear space and p a function $X \to \mathbb{R}$ that satisfies

(i) positive homogeniety: $\forall x \in X, \alpha > 0 : \; p(\alpha x) = \alpha p(x)$, and

(ii) sub-additivity: $\forall x, y \in X : \; p(x + y) \leq p(x) + p(y)$.

Let φ be a linear functional defined on $Y \subseteq X$, where Y is a linear subspace, such that

$$\forall y \in Y : \; \varphi(y) \leq p(y).$$

Then there exists an extension of φ to X such that $\forall x \in X : \; \varphi(x) \leq p(x)$. □

Remark 3.28.

- If p is absolute homogeneous, i.e. $\forall \alpha \in \mathbb{R} : \; p(\alpha x) = |\alpha| p(x)$, then p is a pseudo-norm, i.e. a norm without $\forall x \in X : p(x) = 0 \Rightarrow x = 0$.
- Typically, p is a norm. //

Proof of theorem 3.27 – Part 1/2. Steps:

1. Suppose $Y \neq X$, then there is a $z \in X, z \notin Y$. We aim to define $\varphi(z)$ such that $\varphi \leq p$ on $\text{span}(Y \cup \{z\})$. We need to find $\varphi(z)$ such that $\forall y \in Y, \alpha \in \mathbb{R} : \; \varphi(y + \alpha z) \leq p(y + \alpha z)$. For $\alpha > 0$ we have $p(y + \alpha z) = \alpha p(\frac{y}{\alpha} + z) = \alpha p(y' + z)$, where we have put $y' := \frac{y}{\alpha} \in Y$. We need to verify the cases $\alpha = +1$ and $\alpha = -1$, i.e. $\varphi(y + z) \leq p(y + z)$ and $\varphi(y' - z) \leq p(y' - z)$. We have $\forall y, y' \in Y$:

$$\begin{aligned}
\begin{matrix} \varphi(y) + \varphi(z) \leq p(y + z) \\ \varphi(y') - \varphi(z) \leq p(y' - z) \end{matrix} \quad &\Leftrightarrow \quad \varphi(y') - p(y' - z) \leq \varphi(z) \leq p(y + z) - \varphi(y) \\
&\Leftrightarrow \quad \varphi(y') - p(y' - z) \leq p(y + z) - \varphi(y) \\
&\Leftrightarrow \quad \varphi(y') + \varphi(y) \leq p(y' - z) + p(y + z) \\
&\Leftrightarrow \quad \varphi(y' + y) \leq p(y + y') = p(y + z + y' - z) \leq p(y + z) + p(y' - z) \qquad \checkmark
\end{aligned}$$

2015-05-29

2. Next lecture. ■

Repitition: Hahn-Banach theorem (real version): Let X be a real linear space and $p\colon X \to \mathbb{R}$ satisfiying:

(i) $\forall \alpha > 0: \ p(\alpha x) = \alpha p(x)$

(ii) $p(x+y) \leq p(x) + p(y)$

Let Y be a linear subspace of X and φ a functional on Y such that

$$\forall y \in Y: \ \varphi(y) \leq p(y), \tag{$*$}$$

then there exists an extension of φ to all X such that φ is linear and $\forall x \in X: \ \varphi(x) \leq p(x)$.

Proof of theorem 3.27 – Part 2/2. Steps:

1. For any $z \notin y$, there exists an extension to $\operatorname{span}(Y \cup \{z\})$, such that $(*)$ holds on $\operatorname{span}(Y \cup \{z\})$.
2. Apply Zorn's lemma: Let (W, φ) be a set of all extensions (that satisfy $(*)$), is partially ordered by $(W, \varphi) \preceq (W', \varphi')$ if $W \subseteq W'$ and $\varphi = \varphi'$ on W. All satisfy $W \supseteq Y$ and ϕ in Y is as in the theorem. Let $(W_\alpha, \varphi_\alpha)$ be a linearly ordered subset, then $W := \bigcup_{\alpha \in A} W_\alpha$ and $\varphi(x) = \begin{cases} \varphi_\alpha(x) & \text{for } x \in W_\alpha \end{cases}$. We need to check $\forall \alpha \in A: (W_\alpha, \varphi_\alpha) \prec (W, \varphi)$, but by construction $W_\alpha \subseteq W$ and $\varphi = \varphi_\alpha$ on W_α, so (W, φ) is an upper bound. By virtue of Zorn's lemma, the set of extension has a maximal element. Let $(\tilde{W}, \tilde{\varphi})$ be a maximal element, then $\tilde{W} = X$. ■

Theorem 3.29 *(Hahn-Banach theorem – complex version)***.** Let X be a complex linear space and $p\colon X \to \mathbb{R}$ a pseudo-norm (i.e. change condition 3.27.(i) to $\forall \alpha \in \mathbb{C}: \ p(\alpha x) = |\alpha| p(x)$). Let Y be a linear subspace of X and φ a linear functional on Y such that $\forall y \in Y: \ |\varphi(y)| \leq p(y)$. Then there exists an extension of φ to X such that φ is linear and $\forall x \in X: \ |\varphi(x)| \leq p(x)$. □

Proof. Similar to the proof of the real version. ■

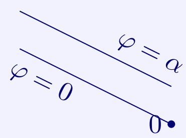
Application of Hahn-Banach theorem:

Lemma 3.30 *(existence of tangent)***.** Let X be a normed linear space and $x_0 \in X$. Then there exists a $\varphi \in X^*$ such that $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$. □

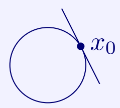
Proof. Let $x_0 \neq 0$, and define $Y = \{\alpha x_0 \mid \alpha \in \mathbb{F}\}$ and $p\colon X \to \mathbb{R}$, $p(x) = \|x\|$. On Y define $\varphi(\alpha x_0) = \alpha \|x_0\|$. Then by Hahn-Banach theorem, there exists a φ on X such that $|\varphi(x)| \leq \|x\|$ and $\varphi(\alpha x_0) = \alpha \|x_0\|$. By construction $\|\varphi\| \leq 1$, but $\varphi(x_0) = \|x_0\|$, and hence $\|\varphi\| = 1$. ■

Definition 3.31 *(hyperplane, half space, tangent)***.** Let X be a real vectorspace.

A subspace $Y \subseteq X$ is called a *hyperplane*, if there exists $\varphi \in X^*$ and $\alpha \in \mathbb{R}$ such that $Y = \{x \in X \mid \varphi(x) = \alpha\} =: \{\varphi = \alpha\}$. Sets $\{x \in X \mid \varphi(x) < \alpha\}$, resp. $\{x \in X \mid \varphi(x) > \alpha\}$ are called *open half spaces*.



A *tangent* to a set K at a point $x_0 \in K$ is a hyperplane $Y = \{\varphi = \alpha\}$ such that $x_0 \in Y$ and $K \subseteq \{\varphi \leq \alpha\}$. Look at $B_1 = \{\|x\| \leq 1\}$. We have any $\|x_0\| = 1$, therefore there exists φ such that $\varphi(x_0) = 1$ and for $x \in B_1$ $\varphi(x) \leq 1$.



Remark 3.32 *(uniqueness in Hahn-Banach theorem)***.** Concering lemma 3.30:

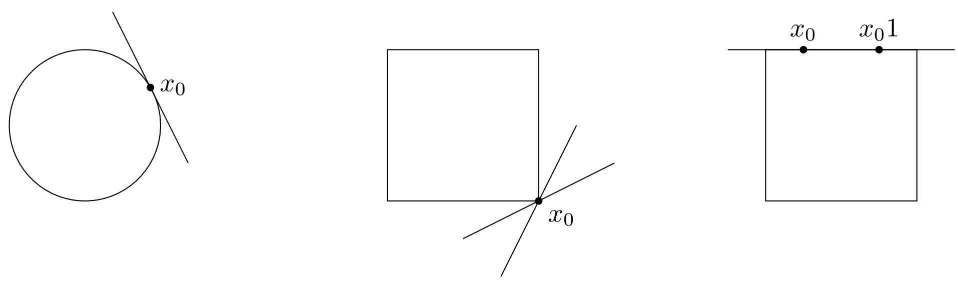


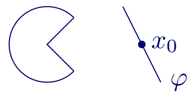
Figure 4: Tangents to subspaces of $\mathbb{R}^2$

Middle figure: At some point there may be more than one tangent.

Right figure: One tangent can be tangent to several points. //

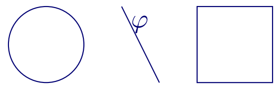
Geometrical versions of Hahn-Banach theorem in real vector spaces:

Theorem 3.33 *(Mazur's theorem)***.** Let X be a real normed linear space.
Let further K be an open convex subset of X, and $x_0 \in X$, $x_0 \notin K$. Then there exists a hyperplane $Y = \{\varphi = \alpha\}$ such that $x_0 \in Y$ and $K \subseteq \{\varphi < \alpha\}$.



□

Theorem 3.34 *(Geometrical Hahn-Banach theorem)***.** Let X be a normed linear space.
Let $K, \tilde{K}$ be two disjoint open convex subsets of normed linear space X. Then there exists $\varphi \in X^*$ and $\alpha \in \mathbb{R}$ such that $\forall y \in K : \varphi(y) < \alpha$ and $\forall \tilde{y} \in \tilde{K} : \varphi(\tilde{y}) > \alpha$.



□

Remark 3.35 *(complex projective space)***.** Look at $\mathbb{C}$, $z = z_0$, $\mathbb{C}^2 \sim (z, w)$. $\varphi(z, w) = (3 + 1)z + w = 0$ (can't read blackboard). $\mathbb{CP} = \{\text{space of all lines in } \mathbb{C}^2\}$. By Poincare duality, $\mathbb{CP} \sim$ sphere in S^3. //

Lemma 3.36 *(dual representation of norm)***.** Let X be a normed linear space. Then, for any $x \in X$

$$\|x\| = \sup_{\varphi \in X^*, \|\varphi\|=1} |\varphi(x)|.$$

□

Proof. $|\varphi(x)| \leq \|\varphi\| \|x\|$, in particular $\sup_{\varphi \in X^*, \|\varphi\|=1} |\varphi(x)| \leq \|x\|$. By existence of tangent, there is a φ such that $|\varphi(x)| = \|x\|$ and $\|\varphi\| = 1$. ■

3.5 Reflexive Spaces

Definition 3.37 *(bidual space, canonical embedding)***.** Let X be a normed linear space and $Y = X^*$, then $Y^* = X^{**}$ is called the *bidual space* of X. By definition X^{**} is a normed linear space and for $\varepsilon \in X^{**}$

$$\|\varepsilon\| = \sup_{\varphi \in X^*, \|\varphi\|=1} |\varepsilon(\varphi)|.$$

Let $x \in X$ and define $J_x \in X^{**}$ by

$$J_x \colon X^* \to \mathbb{F}, \ J_x(\varphi) = \varphi(x).$$

We obtain a map $J \colon X \to X^{**}$, $x \mapsto J_x$, the *canonical embedding*.

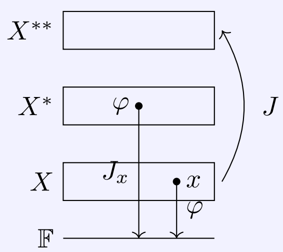


Figure 5: Schematic illustration of the bidual space and the canonical embedding.

Proof that $J_x \in X^{**}$ in definition 3.37.

(1) Linearity: $J_x(\varphi + \alpha\tilde{\varphi}) = (\varphi + \alpha\tilde{\varphi})(x) = \varphi(x) + \alpha\tilde{\varphi}(x) = J_x(\varphi) + \alpha J_x(\tilde{\varphi})$

(2) Boundedness: $\|J_x(\varphi)\| = |\varphi(x)| \leq \|\varphi\| \|x\|$ ■

Theorem 3.38 *(canonical embedding is isometry)***.** The canonical embedding is an isometric isomorphism of $X \to J[X] \subseteq X^{**}$. □

Proof. We only proof the "isometric" part of the claim:

$$\|J_x\| = \sup_{\varphi \in X^*, \|\varphi\|=1} |J_x(\varphi)| = \sup_{\varphi \in X^*, \|\varphi\|=1} |\varphi(x)| = \|x\|.$$

■

Remark 3.39 *(linear isometries are injective)***.** Linear isometries are always injective. //

Definition 3.40 *(reflexive space)***.** Space X is called *reflexive* if J is surjective, i.e. $J[X] = X^{**}$.

Remark 3.41.

- Reflexive spaces are always complete, hence Banach.
- If $\overline{J[X]} \subseteq X^{**}$ (*Remark by the typesetter: this is always true*), then $\overline{J[X]}$ is a Banach space. $\overline{J[X]}$ is a completition of X.
- There exists a space X such that X and X^{**} are isometrically isomorphic, but X is not reflexive.

//

Remark about completitions:

Definition 3.42 *(completition)***.** Let X be a normed linear space. A mapping $\phi\colon X \to Y$ is called *completition* of X, if Y is complete, $\phi[X]$ is dense in Y, and ϕ is an isometric homomorphism. The pair (ϕ, Y) is called *completition* of X.

Example 3.43 *(standard completition)***.** Consider the space of all Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in X and equip it with the equivalence relation

$$[(x_n)_{n\in\mathbb{N}}] = [(\tilde{x}_n)_{n\in\mathbb{N}}] \;\Leftrightarrow\; \lim_{n\to\infty} x_n = \lim_{n\to\infty} \tilde{x}_n.$$

Then put $Y = \{[(x_n)_{n\in\mathbb{N}}] \mid (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \text{ cauchy}\}$. ◊

Prop. 3.44 *(Hilbert spaces are reflexive)***.** All Hilbert spaces are reflexive. □

Proof. Preliminary remark: $X = \mathcal{H}$, $X \cong X^*$ by Riesz duality:

$$\Phi\colon \mathcal{H} \to \mathcal{H}^*,\; \Phi(x) = \varphi_x, \quad \varphi_x(y) = \langle x, y\rangle$$

So

$$(\mathcal{H}^*)^* \overset{\tilde{\Phi}}{\cong} \mathcal{H}^* \overset{\Phi}{\cong} \mathcal{H}.$$

Proof itself: Let Φ be a Riesz duality between $\mathcal{H}$ and $\mathcal{H}^*$. $\mathcal{H}^*$ itself is a Hilbert space, $\langle \varphi_x, \varphi_y\rangle = \langle y, x\rangle$. Then we have a map

$$\tilde{\Phi}\colon \mathcal{H}^* \to \mathcal{H}^{**},\; \varphi_x \mapsto \Phi(\varphi_x) = \varepsilon_{\varphi_x}, \quad \varepsilon_{\varphi_x}(\varphi_y) = \langle \varphi_x, \varphi_y\rangle.$$

We will check that $\tilde{\Phi} \circ \Phi = J$:

$$\big((\tilde{\Phi} \circ \Phi)(x)\big)(\varphi_y) = \big(\tilde{\Phi}(\varphi_x)\big)(\varphi_y) = \varepsilon_{\varphi_x}(\varphi_y) = \langle \varphi_x, \varphi_y\rangle = \langle y, x\rangle = \varphi_y(x) = J_x(\varphi_y) \quad \therefore \quad \tilde{\Phi} \circ \Phi = J$$ ■

Example 3.45 *(examples and counterexamples of reflexive spaces)***.**

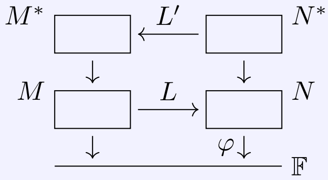
(1) $L^p(X, \Sigma, \mu)$ is reflexive for $p \in]1, \infty[$, in particular ℓ^p is reflexive for $p \in]1, \infty[$.
$(L^p)^* = L^q$, $(L^q)^* = L^p$, $\frac{1}{p} + \frac{1}{q} = 1$.

(2) L^1 and L^∞ are not reflexive.

(3) $c_0, c_1, C([0, 1])$ are not reflexive. ◊

3.6 The Conjugate of an Operator

Definition 3.46 *(Banach conjugate)***.** Let M, N be normed linear spaces and $L \in \mathcal{L}(M, N)$. Then the *Banach conjugate* L' is a linear map $L' \in \mathcal{L}(N^*, M^*)$ defined by $\forall \varphi \in N^*, x \in M : \; (L'(\varphi))(x) = \varphi(L(x))$.



$$L'(\varphi) = \varphi \circ L$$

Figure 6: Schematic illustration of the banach conjugate of a bounded operator.

Prop. 3.47 *(calculation rules for the Banach conjugate)*. Let M, N, P be normed linear spaces and $\alpha \in \mathbb{F}$, $T, L \in \mathcal{L}(M,N)$, $S \in \mathcal{L}(N,P)$. Then we have (recall $S \circ L \in \mathcal{L}(M,P)$):

(i) $\|L'\| = \|L\|$

(ii) $(\alpha \cdot L)' = \alpha \cdot L'$

(iii) $(L+T)' = L' + T'$

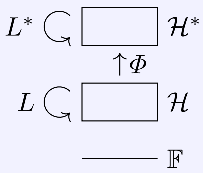
(iv) $(S \circ L)' = L' \circ S'$ □

Proof. Recall that $\forall \varphi \in N^* : \; L'(\varphi) = \varphi \circ L$, so linearity follows. We prove only $\|L'\| = \|L\|$.

$$\forall \varphi \in N^* : \; \|L'(\varphi)\| = \sup_{\substack{x \in M \\ \|x\|=1}} |(L'(\varphi))(x)| = \sup_{\substack{x \in M \\ \|x\|=1}} |\varphi(L(x))|$$

$$\|L'\| = \sup_{\substack{\varphi \in N^* \\ \|\varphi\|=1}} \|L'(\varphi)\| = \sup_{\substack{\varphi \in N^* \\ \|\varphi\|=1}} \sup_{\substack{x \in M \\ \|x\|=1}} |\varphi(L(x))| = \sup_{\substack{x \in M \\ \|x\|=1}} \|L(x)\| = \|L\|$$ ■

Definition 3.48 *(Hermitian conjugate)*. Let $\mathcal{H}$ be a Hilbert space, $L \in \mathcal{L}(\mathcal{H})$ a bounded operator, $L' \in \mathcal{L}(\mathcal{H}^*)$ its Banach conjugate. Then we define $L^* = \Phi^{-1} \circ L' \circ \Phi \in \mathcal{L}(\mathcal{H})$ to be the *Hermitian conjugate* of L.



$$\Phi(x) = \varphi_x$$
$$\varphi_x(y) = \langle x, y \rangle$$

Figure 7: Schematic illustration of the hermitian conjugate of a bounded operator.

Prop. 3.49.

$$\langle x, L(y) \rangle = \langle L^*(x), y \rangle$$ □

Proof.

$$\langle x, L(y) \rangle = (\varphi_x \circ L)(y) = (L'(\Phi(x)))(y) = \left\langle (\Phi^{-1} \circ L' \circ \Phi)(x), y \right\rangle = \langle L^*(x), y \rangle$$ ■

Definition 3.50 *(Hermitian operator)*. An operator $L \in \mathcal{L}(\mathcal{H})$ is called *Hermitian*, if $L^* = L$.

3.7 Compact Operators

Definition 3.51 *(compact operator)*. Let M, N be Banach spaces. A linear operator $L \colon M \to N$ is called *compact*, if it maps bounded sets M to relatively compact sets in N. The space of all compact operators is denoted by $\mathcal{L}_{\text{cpt}}(M,N)$.

Prop. 3.52 *(characterization of compact operators)*. Equivalent definitions of a compact operator:

(i) L maps bounded sets M to relatively compact sets in N.

(ii) For any bounded sequence $(x_n)_{n \in \mathbb{N}}$ the bounded sequence $(Lx_n)_{n \in \mathbb{N}}$ has a convergent subsequence.

(iii) If we denote $B_1 = \{x \in M \mid \|x\| \leq 1\}$, then LB_1 is a relatively compact set. □

Definition 3.53 *(finite-rank operator)***.** A linear operator $L\colon M \to N$ is called *finite-rank* if $L \in \mathcal{L}(M,N)$ and $\operatorname{im}(F)$ is a finite-dimensional space. The space of all finite-rank operators is denoted by $\mathcal{L}_{\mathrm{f}}(M,N)$.

Prop. 3.54 *(properties of $\mathcal{L}_{cpt}(M,N)$)***.** Let M, N be Banach spaces. Then:

(i) $\mathcal{L}_{\mathrm{f}}(M,N) \subseteq \mathcal{L}_{\mathrm{cpt}}(M,N) \subseteq \mathcal{L}(M,N)$.

(ii) If $(L_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}_{\mathrm{cpt}}(M,N)$ and $L_N \overset{N\to\infty}{\longrightarrow} L$, i.e. $\|L_N - L\| \overset{N\to\infty}{\longrightarrow} L$, with $L \in \mathcal{L}(M,N)$, then $L \in \mathcal{L}_{\mathrm{cpt}}(M,N)$. I.e. $\mathcal{L}_{\mathrm{cpt}}(M,N)$ is closed.

(iii) If $L \in \mathcal{L}(M,N), S \in \mathcal{L}(N,P)$, then $S \circ L \in \mathcal{L}(M,P)$ is compact if L or S is compact. I.e. $\mathcal{L}_{\mathrm{cpt}}(M,N)$ is a two-sided ideal in $\mathcal{L}(M,N)$. □

Example 3.55 *(Volterra integral operator is compact)***.** The Volterra integral operator $L\colon C([0,1]) \to C([0,1])$, $(Lf)(x) = \int_0^x K(x,y)\cdot f(y)\,\mathrm{d}y$ is compact. ◊

2015-06-05

Theorem 3.56 *(properties of $\mathcal{L}_{cpt}(M,N)$)***.**

(i) $\mathcal{L}_{\mathrm{f}}(M,N) \subseteq \mathcal{L}_{\mathrm{cpt}}(M,N) \subseteq \mathcal{L}(M,N)$

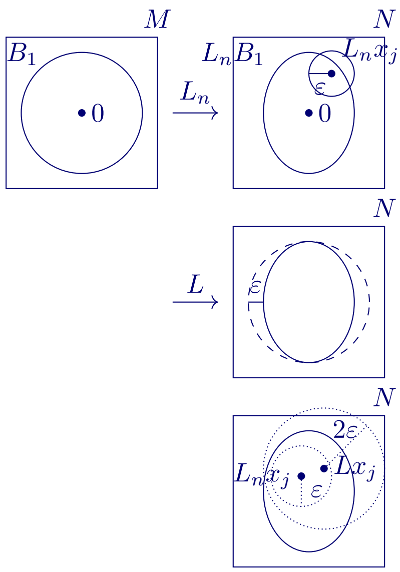
(ii) $\mathcal{L}_{\mathrm{cpt}}(M,N)$ is a closed subspace of $\mathcal{L}(M,N)$

(iii) $\mathcal{L}_{\mathrm{cpt}}(M,N)$ is a two-sided ideal, i.e. for any $T, L \in \mathcal{L}(M,N)$, TL is compact whenever T or L is. □

Proof.

(i) If $L \in \mathcal{L}_{\mathrm{cpt}}(M,N)$ then LB_1 is relatively compact hence bounded.

(ii) We need to prove that if $L_n \in \mathcal{L}_{\mathrm{cpt}}$ and $L_n \overset{n\to\infty}{\longrightarrow} L$, i.e. $\|L_n - L\| \overset{n\to\infty}{\longrightarrow} 0$, then $L \in \mathcal{L}_{\mathrm{cpt}}(M,N)$.
Fix $\varepsilon > 0$. We know L_n is compact, so there are $x_1,\dots,x_k \in B_1$ such that
$$\bigcup_{j=1}^{k} B_\varepsilon(L_n x_j) \supseteq L_n B_1.$$
I can find n large enough such that $\|L_n - L\| \le \varepsilon$. For each x_j we have $\|L_n x_j - Lx_j\| \le \varepsilon$. It follows that
$$\bigcup_{j=1}^{k} B_{2\varepsilon}(Lx_j) \supseteq LB_1.$$



(iii) We want to show that TL is compact.
Case 1: L compact: then LB_1 is relatively compact.
Claim: Bounded operator maps relatively compact sets to relatively compact sets. If $x_n \overset{n\to\infty}{\longrightarrow} x$, then of course $Tx_n \overset{n\to\infty}{\longrightarrow} Tx$.
Case 2: T relatively compact: L maps B_1 into a bounded set.

■

Corollary 3.57. Let $L \in \mathcal{L}_{\mathrm{cpt}}(X)$ be a compact operator in an infinite-dimensional Banach space X. Then the operator does not have a continuous inverse. □

Proof. Inverse map $L^{-1}L = \mathrm{id}$ (then $LL^{-1} = \mathrm{id}$). Suppose that L^{-1} is bounded map. Then by (iii) id is a compact map. Contradiction to theorem 2.15. ■

Question: Does $\overline{\mathcal{L}_{\mathrm{f}}} = \mathcal{L}_{\mathrm{cpt}}$ hold?
Answer: Not always, but often (e.g. in Hilbert spaces).
In the following, we fix $\mathcal{H}$, separable Hilbert space with basis $\{e_n\}_{n=1}^{\infty}$. $\mathcal{L}_{\mathrm{cpt}}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$.

Definition 3.58 *(matrix element)*. For $L \in \mathcal{L}(\mathcal{H})$ we define the (j,k)-th *matrix element* of L as $L_{jk} = \langle e_j, Le_k \rangle$.

Recall chopping infinite systems of linear equations in the introduction:

$$\left(\begin{array}{cccc|c} L_{11} & L_{12} & & & \\ L_{21} & L_{22} & & & \\ & & \ddots & & \\ & & & L_{NN} & \\ \hline & & & & \ddots \end{array}\right) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ \hline \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \\ \hline \vdots \end{pmatrix}$$

If $\overline{\mathcal{L}_\mathrm{f}} = \mathcal{L}_\mathrm{cpt}$, then we can approximate compact operators by finite-rank operators, i.e. the chopping works. But at first, we have to define "chopping" rigorously.

Definition 3.59 *(chopping of operators)*. Define P as orthogonal projection into $\operatorname{span}\{e_1, \dots, e_N\}$:

$$P\left(\sum_{j=1}^{\infty} x_j e_j\right) = \sum_{j=1}^{N} x_j e_j \quad \text{or} \quad P(\cdot) = \sum_{j=1}^{N} e_j \langle e_j, \cdot \rangle$$

"*Chopping*" of L is operator $P_N L P_N$. By definition $P_N L P_N$ is finite rank. Note that also $P_N L$ and $L P_N$ are finite rank.

Concering the matrix elements: Let $x \in \mathcal{H}$, $x = \sum_{j=1}^{\infty} x_j e_j$, $x_j = \langle e_j, x \rangle$. Isometry $\mathcal{L} \leftrightarrow \ell^2$, $x \mapsto (x_j)_{j=1}^{\infty}$.

For a bounded operator L:

$$Lx = L \sum_{j=1}^{\infty} x_j e_j = \sum_{j=1}^{\infty} x_j (Le_j) = \sum_{j=1}^{\infty} x_j \sum_{k=1}^{\infty} e_k \underbrace{\langle e_k, Le_j \rangle}_{=L_{kj}} = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} L_{kj} x_j \right) e_k$$

Projection:

$$P_N(\cdot) := \sum_{n=1}^{\infty} e_n \langle e_n, \cdot \rangle$$

Isometry $\mathcal{L} \leftrightarrow \ell^2$:

$$\begin{aligned} x &\mapsto (x_n)_{n=1}^{\infty} \\ Lx &\mapsto \left(\sum_{j=1}^{\infty} L_{nj} x_j \right)_{n=1}^{\infty} \\ P_N L P_N x &\mapsto \left(\sum_{j=1}^{N} L_{nj} x_j \right)_{n=1}^{N} \quad \text{for } n \le N \\ P_N L P_N x &\mapsto 0 \quad \text{for } n > N \end{aligned}$$

Remark: *Decomposition of identity in Hilbert spaces*:

$$\sum_{n=1}^{\infty} e_n \langle e_n, \cdot \rangle = \mathrm{id}$$

Theorem 3.60 *(approximation of compact operators by finite-rank operators)*. Let $\mathcal{H}$ be a separable Hilbert space and $L \in \mathcal{L}_\mathrm{cpt}(\mathcal{H})$. Then

$$P_N L \overset{N\to\infty}{\longrightarrow} L, \quad L P_N \overset{N\to\infty}{\longrightarrow} L, \quad P_N L P_N \overset{N\to\infty}{\longrightarrow} L..$$

In particular

$$\overline{\mathcal{L}_\mathrm{f}(\mathcal{H})} = \mathcal{L}_\mathrm{cpt}(\mathcal{H}). \qquad \square$$

In order to prove theorem 3.60, we need:

Prop. 3.61 *(characterization of relatively compact sets in Hilbert spaces)*. Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}_{n=1}^{\infty}$ basis.

A bounded set K is relatively compact iff

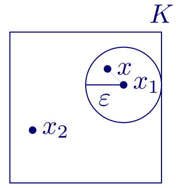
$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall x \in K : \sum_{n=N}^{\infty} \left| \langle e_j, x \rangle \right|^2 < \varepsilon. \qquad \square$$

Remark 3.62 *(Remark to proposition 3.61)***.** Recall Parseval's identity:

$$\sum_{n=1}^{\infty} \left| \langle e_j, x \rangle \right| = \|x\|^2$$

Here in proposition 3.61 in addition, N can be choosen uniformly. //

Proof of proposition 3.61. Direction "⇒":
If K is relatively compact, then there exist $x_1, \dots, x_n$ such that



$$\bigcup_{j=1}^{n} B_\varepsilon(x_j) \supseteq K.$$

By Bessel inequality, there exists a N such that

$$\forall k = 1, \dots, n : \sum_{j=N}^{\infty} \left| \langle e_j, x_k \rangle \right|^2 \leq \varepsilon.$$

Let $x \in K$, then there is a x_j such that $\|x - x_j\| \leq \varepsilon$. Then

$$\sqrt{\sum_{j=N+1}^{\infty} |\langle e_j, x \rangle|^2} \overset{\substack{\text{calculation} \\ \text{as in} \dots}}{=} \|(1 - P_N)x\| = \|(1 - P_N)(x - x_j) + (1 - P_N)x_j\| \leq \|(1 - P_N)(x - x_j)\| + \|(1 - P_N)x_j\| \leq \varepsilon + \sqrt{\varepsilon},$$

where we have used that $\|1 - P_N\| = 1$. ■

Proof of theorem 3.60. Only $\|P_N L - L\| \overset{N \to \infty}{\longrightarrow} 0$. $\|P_N L - L\| = \|(1 - P_N)L\|$. For each $\varepsilon \geq N$ it holds that $\|(1 - P_N)L\| \leq \varepsilon$. Let $K = LB_1$, then $\|(1 - P_N)L\| = \sup_{x \in B_1} \|(1 - P_N)Lx\| = \sup_{x \in K} \|(1 - P_N)x\|$. Furthermore,

$$\|(1 - P_N)x\|^2 = \sum_{n=N+1}^{\infty} |\langle e_n, x \rangle|^2,$$

because if $x = \sum_{n=1}^{\infty} e_n \langle e_n, x \rangle$ then

$$(1 - P_N)x = \sum_{n=N+1}^{\infty} e_n \langle e_n, x \rangle$$

$$\left\| \sum_{n=N+1}^{\infty} e_n \langle e_n, x \rangle \right\|^2 = \sum_{n=N+1}^{\infty} |\langle e_n, x \rangle|^2 \quad \text{(Pythagoras)}.$$

We know that K is relatively compact, and so there exists a N such that $\forall x \in K : \sum_{n=N+1}^{\infty} |\langle e_n, x \rangle|^2 \leq \varepsilon$. We conclude $\|(1 - P_N)L\| \leq \varepsilon$ ■

Remark 3.63. It is $\|\mathrm{id} - P_N\| = 1$. Hope $P_n \overset{N \to \infty}{\longrightarrow} \mathrm{id}$ (but not true in this norm). For each x $\|P_N x - x\| \overset{N \to \infty}{\longrightarrow} 0$. //

3.8 Weak Topology and Weak Convergence

Definition 3.64 *(weak convergence)***.** Let X be normed linear space. We say that $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ *converges weakly* to X,

$$x_n \overset{\mathrm{w}}{\longrightarrow} x,$$

if for all $\varphi \in X^*$ we have

$$\varphi(x_n) \to \varphi(x).$$

Prop. 3.65 *(basic properties of weak convergence)***.**

(1) Weak limit is unique.

(2) If $x_n \longrightarrow x$ then $x_n \xrightarrow{\text{w}} x$. □

Proof.

(1) Suppose $x_n \xrightarrow{\text{w}} x$ and $x_n \xrightarrow{\text{w}} \tilde{x}$. Then for each $\varphi \in X^*$ we have $\varphi(x - \tilde{x}) = 0$. By existence of tangent there is $\varphi \in X^*$ such that $\varphi(x - \tilde{x}) = \|x - \tilde{x}\| = 0$.

(2) $|\varphi(x - x_n)| \leq \|\varphi\| |x - x_n|$ ✓ ■

Definition 3.66 *(weak*-convergence)***.** Let X be a normed linear space and X^* its dual space. We say that for $(\varphi_n)_{n \in \mathbb{N}} \in (X^*)^{\mathbb{N}}$

$$\varphi_n \xrightarrow{\text{w}^*} \varphi,$$

if for all $x \in X$ we have

$$\varphi_n(x) \longrightarrow \varphi(x).$$

Remark 3.67 *(illustration of weak and weak* convergence)***.** Recall the canonical embedding $J\colon x \mapsto \varepsilon_x$ where $\varepsilon_x(\varphi) = \varphi(x)$.

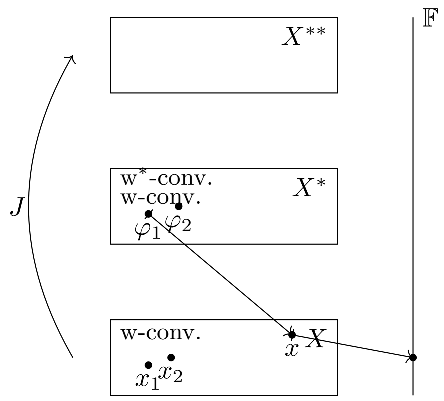


Figure 8: illustration of weak and weak* convergence

//

Prop. 3.68 *(basic properties of weak* convergence)***.**

(a) Weak*-limit is unique

(b) If $\varphi_n \xrightarrow{\text{w}} \varphi$ then $\varphi_n \xrightarrow{\text{w}^*} \varphi$. □

Proof.

(a) Omitted.

(b) Suppose that $\varphi_n \xrightarrow{\text{w}} \varphi$. For all $\varepsilon \in X^{**}$, $\varepsilon(\varphi_n) \longrightarrow \varepsilon(\varphi)$. We know that for each $x \in X$, we have

$$\varphi_n(x) = \varepsilon_x(\varphi_n) \longrightarrow \varepsilon_x(\varphi) = \varphi(x),$$

and hence $\varphi_n \xrightarrow{\text{w}^*} \varphi$. ■

Prop. 3.69 *(weak and weak* convergence in reflexive spaces)***.** If X is reflexive, then notions of weak convergence and weak*-convergence coincide. □

Proof. Let $\varphi_n \xrightarrow{\text{w}^*} \varphi$. We know that for all $\varepsilon \in X^{**}$, there exists $x \in X$ such that $\varepsilon = \varepsilon_x$. Then

$$\varepsilon(\varphi_n) = \varphi_n(x) \longrightarrow \varphi(x) = \varepsilon(\varphi),$$

and hence $\varphi_n \xrightarrow{\text{w}} \varphi$. ■

Example 3.70.

(1) Consider $X = c_0$, $X^* = \ell^1$, $X^{**} = \ell^\infty$.
Note ${c_0}^* = \ell^1$: For each $\varphi \in {c_0}^*$ there exists a unique $y \in \ell^1$ such that $\forall x \in c_0 : \ \varphi(x) = \sum_{n=1}^\infty y_n x_n$.
Consider sequence

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \dots), \\ e_2 &= (0, 1, 0, 0, \dots), \\ e_3 &= (0, 0, 1, 0, \dots), \dots. \end{aligned}$$

Claim:

(a) e_n does converge weak*ly, $e_n \xrightarrow{\text{w}^*} 0$.

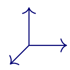
(b) e_n does not converge weakly.

Proof:

(a) For each $x \in c_0$ we need to check that $e_n(x) = \sum_{j=1}^\infty (e_n)_j x_j = x_n$. Then it follows that $\lim_{n\to\infty} e_n(x) = \lim_{n\to\infty} x_n = 0 = 0(x)$.

(b) We have $(\ell^1)^* = \ell^\infty$. Let's take $y = (1, 1, 1, \dots) \in \ell^\infty$. Then $y(e_n) = \sum_{j=1}^\infty y_j (e_n)_j = 1$.

(2) Consider an arbitrary Hilbert space $\mathcal{H}$.



Claim: Let $\{e_n\}_{n=1}^\infty$ be an orthonormal set in $\mathcal{H}$, then $e_n \xrightarrow{\text{w}} 0$.
Proof: By Riesz duality, for each $\varphi \in \mathcal{H}^*$ there exists $y \in \mathcal{H}$ such that

$$\varphi(x) = \langle y, x \rangle.$$

Hence we need to check that for all $y \in \mathcal{H}$ each $\langle y, e_n \rangle \longrightarrow 0$. Bessel's inequality:

$$\sum_{n=1}^\infty |\langle y, e_n \rangle|^2 \le \|y\|^2$$

The sum is convergent, and hence for all $n \in \mathbb{N}$ each $|\langle y, e_n \rangle| \longrightarrow 0$. This proves the claim.

(3) Let $f \in L^2(\mathbb{R})$ and $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}^\mathbb{N}$ such that $t_n \longrightarrow \infty$, and consider $f_n(x) := f(x - t_n)$.
Claim: $f_n \xrightarrow{\text{w}} 0$.
Proof: We need to prove that for each $g \in L^2(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} g(x) \cdot f(x - t_n)\,\mathrm{d}x \longrightarrow 0.$$

We calculate:

$$\begin{aligned} &\left| \int_{-\infty}^{t_n/2} g(x) \cdot f(x - t_n)\,\mathrm{d}x + \int_{t_n/2}^{+\infty} g(x) \cdot f(x - t_n)\,\mathrm{d}x \right| \\ &\le \sqrt{\int_{-\infty}^{t_n/2} g(x)^2\,\mathrm{d}x} \cdot \sqrt{\int_{-\infty}^{t_n/2} f(x - t_n)^2\,\mathrm{d}x} + \sqrt{\int_{t_n/2}^{+\infty} g(x)^2\,\mathrm{d}x \cdot \int_{t_n/2}^{+\infty} f(x - t_n)^2\,\mathrm{d}x} \\ &\longrightarrow 0 \end{aligned}$$

because, by dominated convergence theorem:

$$\int_{-\infty}^{+t_n/2} f(x - t_n)^2\,\mathrm{d}x = \int_{-\infty}^{-t_n/2} f(x)^2\,\mathrm{d}x \longrightarrow 0$$

Illustration: Shifting the function to infinity:

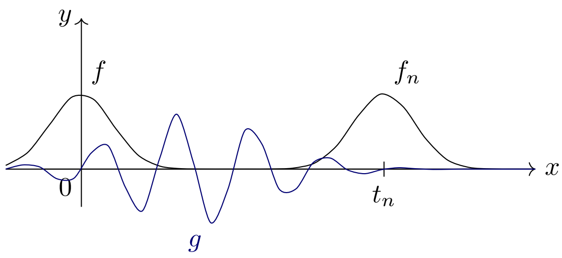


Figure 9: illustration for the proof of example 3.70.(3)

(4) Let $X = C([0,1])$. Then $f_n \xrightarrow{\text{w}} 0$ iff the f_n's are uniformly bounded and $\forall x \in [0,1]: \ f_n(x) \longrightarrow 0$. $\Diamond$

Remark 3.71 *(concentration compactness principle)*. What does it mean $x_n \xrightarrow{\text{w}} 0$ if $\|x_n\| = 1$.

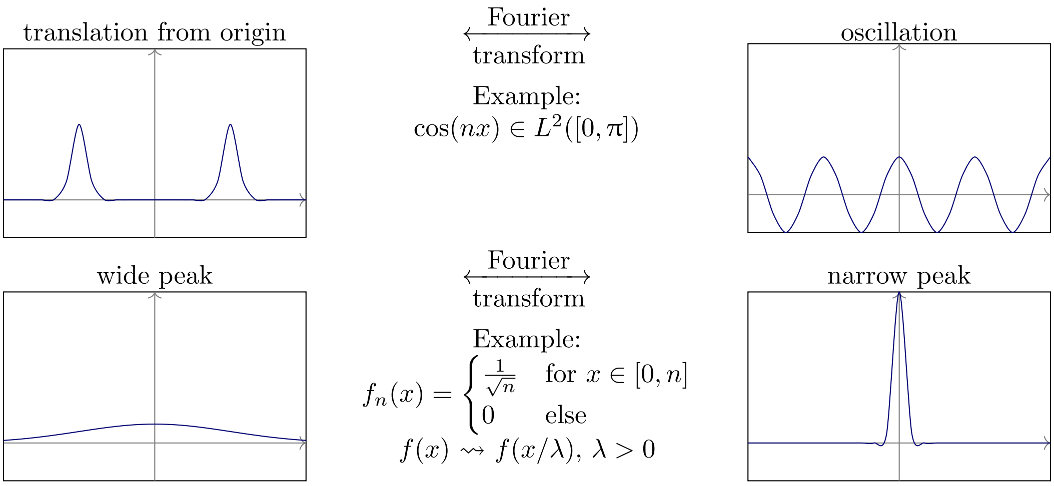


Figure 10: concentration compactness principle

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Remark 3.72 *(the dual space of the space of continuous functions)*. We have:

$$(\{\text{space of continuous functions on } [0,1]\})^* = \text{space of Borel measures on } [0,1]$$

Denote $X = C([0,1])$. If $\varphi \in X^*$, then $\varphi(f) = \int_0^1 f(x) \,\mathrm{d}\mu_x$. Example $\mu_x = \delta(x)$ and $\varphi_x(f) = f(x)$. //

2015-06-12

Question: $(X, \mathcal{T})$ topological space. Suppose $\mathcal{T}$ has more (open) sets then

(A) there are more contiunuous functions $X \to \mathbb{R}$ and less compact sets on X.

(B) there are more contiunuous functions $X \to \mathbb{R}$ and more compact sets on X.

(C) there are less contiunuous functions $X \to \mathbb{R}$ and less compact sets on X.

(D) there are less contiunuous functions $X \to \mathbb{R}$ and more compact sets on X.

Recall:

- A function $f\colon (X, \mathcal{T}) \to \mathbb{R}$ is continuous if $f^{-1}[]a,b[]$ is open
- A set K is compact iff each cover by open sets has a finite subcover

Answer: The correct answer is (A).

Prop. 3.73 *(continuous functions map compact sets to compact sets)***.** Continuous functions on a compact set achieves its minimum and maximum. □

Weak topology:

- $(X, \mathcal{T})$ is a topological space
- We require that functions in X^* are continuous. This means that $\varphi \in X^*$, then you neeed that

$$\varphi^{-1}]a, b[\text{ open } \Leftrightarrow \{x \in X \mid a < \varphi(x) < b\} \text{ open.}$$

Definition 3.74 *(weak topology)***.** The *weak topology* is generated by finite intersections and unions of sets

$$\{x \mid a < |\varphi(x)| < b\}.$$

A set U is weakly open if for each $x \in U$ there exists $\varphi_1, \dots, \varphi_n \in X^*$ and $\varepsilon > 0$ such that

$$\tilde{U}_X := \{y \in X \mid \forall j = 1, \dots, n : |\varphi_j(x) - \varphi_j(y)| < \varepsilon\} \subseteq U.$$

Prop. 3.75 *(convergence in weak topology = weak convergence)***.** A sequence $(x_n)_{n\in\mathbb{N}}$ converges to x w.r.t. the weak topology, if and only if $x_n \xrightarrow{\mathrm{w}} x$. □

Proof. Proof of "⇒": For each open set $U \ni x$ there exists n_0 such that $\forall n \geq n_0 : \; x_n \in U$. We need to show that $x_n \xrightarrow{\mathrm{w}} x$, i.e. $\forall \varphi \in X^* : \; \varphi(x_n) \longrightarrow \varphi(x)$. Let $\varepsilon > 0$. In particular, $U_x = \{y \mid |\varphi(x) - \varphi(y)| < \varepsilon\}$ is open, so there exists n_0 such that for $n > n_0$ we have $x_n \in U_x$, hence $|\varphi(x_0) - \varphi(x)| < \varepsilon$. We conclude $\varphi(x_n) \longrightarrow \varphi$.
Proof of "⇐": See lecture notes. ■

Remark 3.76. Set of weakly converging sequences does *not* define weak topology. There are spaces where convergence weak convergence conincide, but not topology and weak topology. //

Example 3.77 *(Schur's lemma)***.** A sequence $(x_n)_{n\in\mathbb{N}} \in \ell^1$ converges weakly iff it converges in $\|\cdot\|_1$-norm. ◊

Lemma 3.78. Let X be an infinite-dimensional normed linear space. And let U be an weakly open set containinig 0. Then there exists a closed non-zero subspace M such that $M \subseteq U$. In particular U is unbounded. □

Proof. There exists $\varphi_1, \dots, \varphi_n$ and $\varepsilon > 0$ such that

$$\tilde{U} = \{x \mid |\varphi_j(x)| < \varepsilon\} \subseteq U.$$

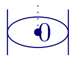
We claim that

$$M = \bigcap_{j=1}^{n} \ker(\varphi_j) \subseteq \tilde{U}$$

is non-zero (in the sense of $M \neq \{0\}$) closed subspace. Suppose that $M = \{0\}$, then the map

$$L \colon X \to \mathbb{F}^n, \; x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$$

is injective (suppose that $Lx = L\tilde{x}$, then $L(x - \tilde{x}) = 0$, hence $(\varphi_1(x - \tilde{x}), \dots, \varphi_n(x - \tilde{x})) = (0, \dots, 0)$, contradiction because there is no injective map infin.-dim. space → finite-dim. space). ■

Remark 3.79. $x_n \xrightarrow{\mathrm{w}} 0$, $\|x_n\| = 1$  //

Definition 3.80 *(weak* topology)***.** Let X^* be the dual of X. The weak* topology on X^* is generated by unions and finite intersections of

$$\{\varphi \mid a < |\varphi(x)| < b\}, \; x \in X, \; a, b > 0.$$

In particular $U \subseteq X^*$ is weak*-open if for each $\varphi \in U$ exists $x_1, \dots, x_n \in X$ and $\varepsilon > 0$ such that

$$\{\psi \mid |\psi(x_j) - \varphi(x_j)| < \varepsilon\} \subseteq U.$$

Prop. 3.81.

(a) If $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in weak* topology, then $\varphi_n \xrightarrow{w^*} \varphi$.

(b) It is the weakest topology on X^* in which functions in $J[X]$ are continuous, where J denoted the canonical embedding.

(c) If X is reflexive, then weak topology on X^* and weak* topology on X^* conincide.

$\square$

Remark 3.82.

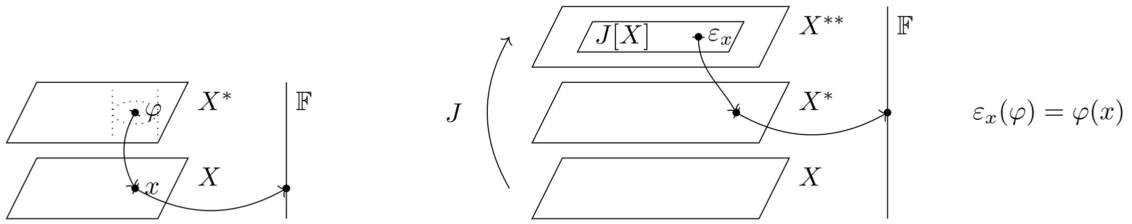


Figure 11: Illustration of dual space and canonical embedding.

//

Theorems 4

2015-06-12

Where we are now?

- Landscape: c, c_0, ℓ^p, L^p, $C([0,1])$, $C^1([0,1])$, …
- Notions: Banach space, norm, compactness, linear operator, …

Now, we're going towards the deep theorems of functional analysis.

4.1 Alaoglu Theorem and its Corollaries

Remark 4.1. Recall $e_1 = (1,0,0,\dots),\ e_2 = (0,1,0,\dots),\ \dots$ in ℓ^1 then $e_n \xrightarrow{w^*} 0$ but $e_n \not\xrightarrow{w} 0$. //

Theorem 4.2 *(Alaoglu theorem)***.** Let X be a Banach space. Then the closed unit ball in X^* is weak* compact. □

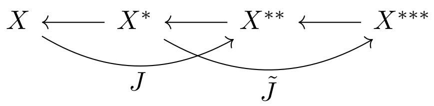
Proof. Omitted. ■

Theorem 4.3 *(Banach-Bourbaki theorem)***.** Let X be a Banach space. Then the closed unit Ball is weakly compact iff X is reflexive. □

Proof.

- Proof of "X reflexive ⇒ unit ball in X weakly compact":

  Situation:

  Claims:

  (C1) If X is reflexive, then J is a homoeomorphism $(X, \text{weak top.}) \to (X^{**}, \text{weak}^* \text{ top.})$

  (C2) X is reflexive iff X^* is reflexive.

  Proofs:

  – Proof of (C2) in direction "⇒":
  If $\alpha \in X^{***}$ then $\alpha \circ J \in X^*$. We will show $\tilde{J}(\alpha \circ J) = \alpha$. $\varepsilon \in X^{**}$, each $\varepsilon = \varepsilon_x = J_x$.

  $$\tilde{J}(\alpha \circ J)(\varepsilon) = \tilde{J}(\alpha \circ J)(\varepsilon_x) = \varepsilon_x(\alpha \circ J) = (\alpha \circ J)(x) = \alpha(\varepsilon_x) = \alpha(\varepsilon)$$

  – Proof of (C2) in direction "⇒":
  We don't need this direction here.

- Proof of "unit ball in X weakly compact ⇒ X reflexive": Omitted.

■

2015-06-16

Repitition:

Theorem 4.2 *(Alaoglu theorem)***.** A unit closed ball in a dual space of a Banach space X is weak* compact. □

Theorem 4.3 *(Banach-Bourbaki theorem)***.** Suppose X is reflexive. Then $\overline{B_1(x)}$ is weakly compact. □

4.2 [Digression] Existence of Solutions to Partial Differential Equations

Example 4.4 *(heat equation)***.** Heat equation:

$$-\Delta u + u = f \quad \text{where} \quad f \in C_0^\infty(\mathbb{R}^d) \quad \text{and} \quad u \in L^2(\mathbb{R}^d). \tag{$*$}$$

Repitition:

- Laplace operator Δ: $\Delta u = \sum_{j=1}^d \frac{\partial^2}{\partial {x_j}^2} u$

- Gradient operator ∇: $(\frac{\partial u}{\partial x_1}u, \ldots, \frac{\partial}{\partial x_d}u)$

Applications:

- This describes heat distribution in a room.
- Similar differential equation for *Black-Scholes equation* which models prices on the stock market.

Remark:

- We skip technicalities (e.g. we require $u \in L^2(\mathbb{R}^d)$, although the consider Δu, it would be more correct to use Sobolev spaces.)

How can we solve $(*)$? ◊

Steps to solve the heat equation

1. Rewrite the equation as minimization problem.

$$\min_{v \in L^2(\mathbb{R}^d)} F(v), \quad F\colon L^2(\mathbb{R}^d) \to \mathbb{R}$$

   Spoiler: Using the Dirichlet principle, we will find:

$$F(v) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} v(x)^2 \,\mathrm{d}x - \int_{\mathbb{R}^d} f(x) \cdot v(x) \,\mathrm{d}x$$

2. Prove that F is bounded from below and weakly lower semi-continuous.
3. Use Banach-Bourbaki to conclude that F achieves its minimum.

1st step to solve the heat equation

Lemma 4.5 *(Dirichlet principle)*. Let

$$F(u) := \int_{\mathbb{R}^d} |\nabla u(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} u(x)^2 \,\mathrm{d}x - \int_{\mathbb{R}^d} f(x) \cdot u(x) \,\mathrm{d}x\,,$$

provided the integrals exist, otherwise $F(\omega) := \infty$. Suppose u is such that $F(u) < \infty$ and $F(u) = \inf_v F(v)$, then u solves $(*)$. □

Proof. Let $g \in C_0^\infty(\mathbb{R}^d)$ and let define $\tilde{F}\colon \mathbb{R} \to \mathbb{R},\ \tilde{F}(\lambda) := F(u + \lambda g)$, then $\forall \lambda \in \mathbb{R}:\ \tilde{F}(0) \leq \tilde{F}(\lambda)$. We calculate

$$\begin{aligned}
\tilde{F}(\lambda) = &\left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 \,\mathrm{d}x + 2\lambda \int_{\mathbb{R}^d} \nabla w(x) \cdot \nabla g(x) \,\mathrm{d}x + \lambda^2 \int_{\mathbb{R}^d} |\nabla g(x)|^2 \,\mathrm{d}x\right) \\
&+ \frac{1}{2}\left(\int_{\mathbb{R}^d} u(x)^2 \,\mathrm{d}x + 2\lambda \int_{\mathbb{R}^d} w(x) \cdot g(x) \,\mathrm{d}x + \lambda^2 \int_{\mathbb{R}^d} g(x)^2 \,\mathrm{d}x\right) \\
&- \left(\int_{\mathbb{R}^d} f(x) \cdot w(x) \,\mathrm{d}x + \lambda \int_{\mathbb{R}^d} f(x) \cdot g(x) \,\mathrm{d}x\right),
\end{aligned}$$

where we have used that

$$|\nabla w + \lambda \nabla g|^2 = \langle \nabla w + \lambda \nabla g, \nabla w + \lambda \nabla g\rangle = |\nabla w|^2 + 2\lambda \langle \nabla w, \nabla g\rangle + \lambda^2 |\nabla g|^2.$$

We note that $\tilde{F}$ is a quadratic form in λ, and because 0 minimizes $\tilde{F}$, we have $\tilde{F}'(0) = 0$.

$$\begin{aligned}
\tilde{F}'(0) = 0 \quad &\Leftrightarrow \quad 2\int_{\mathbb{R}^d} \nabla w(x) \cdot \nabla g(x) \,\mathrm{d}x + \int_{\mathbb{R}^d} u(x) \cdot g(x) \,\mathrm{d}x - \int_{\mathbb{R}^d} f(x) \cdot g(x) \,\mathrm{d}x = 0 \\
&\overset{(*)}{\Leftrightarrow} \quad 2\int_{\mathbb{R}^d} -\Delta w(x) \cdot g(x) \,\mathrm{d}x + \int_{\mathbb{R}^d} w(x) \cdot g(x) \,\mathrm{d}x - \int_{\mathbb{R}^d} (-2\Delta u + u - f) \cdot g(x) \,\mathrm{d}x \\
&\Leftrightarrow \quad -\Delta w(x) + u(x) - f(x) = 0
\end{aligned}$$

step at $(*)$: multivariable version of intgegration by parts = stokes theorem / green identity.
Division by factor 2 yields the claim. ■

3rd step to solve the heat equation

Definition 4.6 *(lower semi-continuity)***.** Function $F\colon X \to \mathbb{R}$ on a topological space X is lower semi-continuous if for all $\alpha \in \mathbb{R}$ the set $\{x \in X \mid F(x) > \alpha\}$ is open, or equivalently, if $x_\alpha \to x$ implies $F(x) \leq \liminf_{x_\alpha \to x} F(x_\alpha)$.

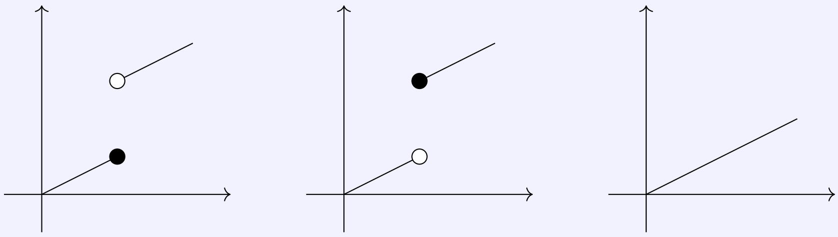


Figure 12: Example of lower semi-continuous (left), upper semi-continuous function (middle), and continuous function (right).

Lemma 4.7. A lower semi-continuous functions achieves its minimum on a compact set. □

Proof. We assume compactness $\Leftrightarrow$ sequential compactness. Let $m := \inf_{x\in K} F(x)$. Let $(x_\alpha)_\alpha$ be a sequence in K such that $F(x_\alpha) \to m$. Because K is compact, there exists a subsequence $x_{\alpha_n} \to x \in K$. Then $m \leq F(x) \leq \liminf_{x_\alpha \to x} F(x_\alpha) = m$, and hence $F(x) = m$. ■

Consequence:

Lemma 4.8. Let X be a reflexive Banach space and $F\colon X \to \mathbb{R}$ a function. Assume:

(i) $\exists \alpha \in \mathbb{R}:\ \{x \in X \mid F(x) \leq \alpha\}$ bounded

(ii) F weakly lower semi-continuous

Then F achieves its infimum on X. □

Proof. The set $\{x \in X \mid F(x) \leq \alpha\}$ is bounded and weakly closed, hence by Banach-Bourbaki it is weakly compact. Then by lemma above, it achieves minimum m on $\{x \in X \mid F(x) \leq \alpha\}$, and therefore $m \leq \alpha$, so it is also a minimum on X. ■

Lemma 4.9. Let X be a Banach space, then $\|\cdot\|$ is weakly lower semi-continuous. □

Proof. Exercise. ■

2nd step to solve the heat equation

Check conditions: Let $\alpha > 0$.

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla v(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int v(x)^2 \,\mathrm{d}x - \int f(x)\cdot v(x)\,\mathrm{d}x \leq \alpha$$

$$\text{LHS} = \frac{1}{2}\int |\nabla v(x)|^2 \,\mathrm{d}x + \frac{1}{2}\int v(x)^2 \,\mathrm{d}x - \int f(x)\cdot v(x)\,\mathrm{d}x \overset{\text{CS}\neq}{\geq} \frac{1}{2}\|v\|^2 - \sqrt{\int f(x)^2\,\mathrm{d}x}\sqrt{\int v(x)^2} = \frac{1}{2}\|v\|^2 - \|f\|\|v\|$$

Therefore:

$$\frac{1}{2}\|v\|^2 - \|f\|\|v\| \leq \alpha$$

So property (i) follows.

$$F(v) = \frac{1}{2}\int \|\nabla v(x)\|^2 \,\mathrm{d}x + \frac{1}{2}\|v\|^2 - \underbrace{\int f(x)\cdot v(x)\,\mathrm{d}x}_{\text{weakly continuous}}$$

Claim:

$$\|\cdot\| \text{ is weakly continuous}$$

Proof: See lemma above.

Claim:

$$\int_{\mathbb{R}^d} |\nabla v(x)|^2 \,\mathrm{d}x \ \text{ is weakly semi-continuous}$$

Proof: Let $v_\alpha \xrightarrow{\mathrm{w}} v$ where $v_\alpha \in C_0^\infty(\mathbb{R}^d)$. I need to compute $\liminf_{v_\alpha \to v} \int_{\mathbb{R}^d} |\nabla v_\alpha(x)|^2 \,\mathrm{d}x$.

$$\begin{aligned}
\big\| |\nabla v_\alpha|^2 \big\|^2 &= \int_{\mathbb{R}^d} |\nabla v_\alpha(x)|^2 \\
&= \sup_{\substack{g \in C_0^\infty(\mathbb{R}^d) \\ \|g\|=1}} \left| \int_{\mathbb{R}^d} g(x) \cdot \nabla v_\alpha(x) \,\mathrm{d}x \right| \\
&= \sup_{\substack{g \in C_0^\infty(\mathbb{R}^d) \\ \|g\|=1}} \left| - \int_{\mathbb{R}^d} \nabla g(x) \cdot v_\alpha(x) \,\mathrm{d}x \right|
\end{aligned}$$

$$\begin{aligned}
\liminf_{v_\alpha \to v} \big\| |\nabla v_\alpha|^2 \big\| &= \liminf_{v_\alpha \to v} \sup_{\substack{g \in C_0^\infty(\mathbb{R}^d) \\ \|g\|=1}} \left| \int_{\mathbb{R}^d} \nabla g(x) \cdot v_\alpha(x) \,\mathrm{d}x \right| \\
&\le \sup_{\substack{g \in C_0^\infty(\mathbb{R}^d) \\ \|g\|=1}} \left| \int_{\mathbb{R}^d} \nabla g(x) \cdot v_\alpha(x) \,\mathrm{d}x \right| \\
&\le \sup_{\substack{g \in C_0^\infty(\mathbb{R}^d) \\ \|g\|=1}} \left| \int_{\mathbb{R}^d} g(x) \cdot \nabla v_\alpha(x) \,\mathrm{d}x \right| \\
&\le \sup_{\substack{g \in C_0^\infty(\mathbb{R}^d) \\ \|g\|=1}} \sqrt{\int g(x)^2 \,\mathrm{d}x} \sqrt{\int |\nabla v(x)|^2 \,\mathrm{d}x} \\
&\le \big\| |\nabla v|^2 \big\|
\end{aligned}$$

Claim:

$$\text{A function } v \mapsto \int_{\mathbb{R}^d} |\nabla v(x)|^2 \,\mathrm{d}x \text{ is weakly lower semi-continuous on } L^2(\mathbb{R}^d).$$

2015-06-19

Let $v_\alpha \xrightarrow{\mathrm{w}} v$, $v_\alpha \in C_0^\infty(\mathbb{R}^d)$.

$$\begin{aligned}
\liminf_{v_\alpha \to v} \sqrt{\int_{\mathbb{R}^d} |\nabla v_\alpha(x)| \,\mathrm{d}x} &= \liminf_{v_\alpha \to v} \|\nabla v_\alpha\| \\
&= \liminf_{v_\alpha \to v} \sup_{f \in C_0^\infty, \|f\|=1} |\langle f, \nabla v_\alpha \rangle| \\
&= \liminf_{v_\alpha \to v} \sup_{f \in C_0^\infty, \|f\|=1} |\langle \nabla f, v_\alpha \rangle| \\
&\ge \sup_{f \in C_0^\infty, \|f\|=1} \liminf_{v_\alpha \to v} |\langle \nabla f, v_\alpha \rangle| \\
&= \sup_{f \in C_0^\infty, \|f\|=1} |\langle \nabla f, v \rangle| \\
&= \|\nabla v\|
\end{aligned}$$

Where we have used that:

$$\langle \nabla f, v_\alpha \rangle = \int_{\mathbb{R}^d} \nabla f(x) \cdot v_\alpha(x) \,\mathrm{d}x = \sum_j \int_{\mathbb{R}^d} f_j(x) \cdot \frac{\partial v_\alpha}{\partial x_j}(x) \,\mathrm{d}x = -\sum_j \int_{\mathbb{R}^d} \frac{\partial f_j}{\partial x_j}(x) \cdot v_\alpha(x) \,\mathrm{d}x = -\int_{\mathbb{R}^d} \nabla f(x) \cdot v_\alpha(x) \,\mathrm{d}x$$

Note that:

$$\liminf_{x_\alpha \to x} = \inf \text{cluster points}$$

$$\inf_{x \in X} \sup_{y \in Y} F(x,y) \ge \sup_{y \in Y} \inf_{x \in X} F(x,y)$$

Conclude:

$$\forall y \in Y: \ \text{LHS} \ge \inf_{x \in X} F(x,y) \quad \therefore \quad \text{LHS} \ge \sup_{y \in Y} \inf_{x \in X} F(x,y)$$

4.3 Baire Category Theorem and its Corollaries

2015-06-1

> *Question:* Let X, Y be normed linear spaces and $L\colon X \to Y$ be a linear operator. Suppose that there exists a ball $B_\varepsilon(z)$ in X such that $L[B_\varepsilon(z)]$ is a bounded set in Y. Is L then a bounded map?

Prop. 4.10. Let X, Y be normed linear spaces and $L\colon X \to Y$ be a linear operator. Suppose that there exists a ball $B_\varepsilon(z)$ in X such that $L[B_\varepsilon(z)]$ is a bounded set in Y. Then L is a bounded map? □

Proof. We have $B_\varepsilon(z) = z + B_\varepsilon(0)$, and so $L[B_\varepsilon(0)] = B_\varepsilon(z) - Lz$ is bounded, and $B_1(0) = \frac{1}{\varepsilon} L[B_\varepsilon(0)]$ is bounded set. Let $x \in B_1(0)$, then $y = z + \varepsilon \in B_\varepsilon(z)$. Then, if $\forall y \in B_\varepsilon(z):\ \|Ly\| \leq M$, we have

$$\|Lx\| = \left\| L\frac{y-z}{\varepsilon} \right\| \leq \frac{1}{\varepsilon} \cdot (\|Ly\| + \|Lz\|) \leq \frac{2M}{\varepsilon}.$$ ■

Definition 4.11 *(interior and closure)*. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ a subset. We define:

$$\text{interior of } A: \quad \operatorname{int}(A) = \bigcup_{B \text{ open with } B \subseteq A} B$$

$$\text{closure of } A: \quad \operatorname{cl}(A) = \bigcap_{B \text{ closed with } B \supseteq A} B$$

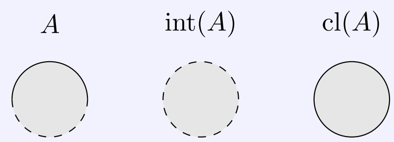


Figure 13: Interior and closure of a subset of a topological space.

Definition 4.12 *(nowhere dense)*. A set A is called nowhere dense if $\operatorname{int}(\operatorname{cl}(A)) = \emptyset$.

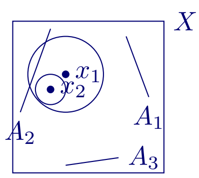
Theorem 4.13 *(Baire category theorem)*. A Banach space X cannot be a countable union of nowhere dense sets. □

Proof. By contradiction.

- Let $x_1 \notin \overline{A_1}$ and $B_{r_1}(x)$ be a small ball such that $\overline{B_{r_1}(x_1)} \cap \overline{A_1} = \emptyset$ and $r_1 < 1$.
- Let $x_2 \in B_{r_2}(x_1)$ and $B_{r_2}(x_2)$ such that $\overline{B_{r_1}} \supseteq B_{r_2}$ and $B_{r_2} \cap \overline{A_2} = \emptyset$ and $r_2 < \frac{1}{2}$.
- Inductively: x_n and $B_{r_n}(x_n)$ such that $\overline{B_{r_n}} \subseteq B_{r_{n-1}}$ and $B_{r_n} \cap \overline{A_n} = \emptyset$ and $r_n < \frac{1}{2^n}$.

Let $m, n > N$, then $x_m, x_n \in B_{r_N}(x_N)$,

$$\|x_n - x_m\| \leq \|x_n - x_N\| + \|x_m - x_N\| \leq \frac{2}{2^N},$$



therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and hence x_n convergent to a point x, $x \overset{n\to\infty}{\longrightarrow} x$. On the other hand, x_n for $n > N$ is such that $\operatorname{dist}(x_n, \overline{A_n}) > \varepsilon > 0$, and therefore $x \notin A_n$ for any N. Contradicition with $X = \bigcup_n A_n$. ■

Remark 4.14 *(categories)*. Why category? A set A is called first category, if A is a countable union of nowhere dense sets. Anything else is second category. //

Remark 4.15. Algebraic or Hamel basis on X. (If X is infinite dimensional Banach space, the Hamel basis is uncountable). //

Example 4.16. Let A be a set of functions in $C([0,1])$ such that $f \in A$ if there is $x \in X$ such that f is differentiable at x. Then A is a set of first category, and therefore there exists $f \in C([0,1])$ such that f is nowhere differentiable. ◊

Theorem 4.17 *(uniform boundedness principle)*. Let X be a Banach space and Y be a normed linear space. Let $\mathcal{F} \subseteq \mathcal{L}(X, Y)$. Then the following is equivalent:

(i) pointwise bound: $\forall x \in X:\ \sup_{L\in\mathcal{F}} \|Lx\| < \infty$

(ii) uniform bound: $\sup_{L\in\mathcal{F}}\|L\| < \infty$ □

Proof. "(ii) ⇒ (i)": $\|Lx\| \le \|L\|\|x\|$. "(i) ⇒ (ii)":
Let

$$A_n := \{x \in X \mid \forall L \in \mathcal{F}:\ \|Lx\| \le n\} = \bigcap_{L\in\mathcal{F}} \{x \in X \mid \|Lx\| \le n\}.$$

Claim (i), then $X = \bigcup_{n\in\mathbb{N}} A_n$, and hence by the Baire category theorem there exists N such that $\overline{A_N}$ has non-empty interior. Then there exists $z \in \overline{A_N}$ and $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq \overline{A_N}$. Therefore $L[B_\varepsilon(z)]$ is bounded for all $y \in B_\varepsilon(z)$, i.e. $\|Ly\| \le N$. If for all $L \in \mathcal{F}$ it holds that $\forall y \in B_\varepsilon(z):\ \|Ly\| \le N$, then it follows that for all $L \in \mathcal{F}$ we have $\|L\| \le \frac{2N}{\varepsilon}$. So, for all $L \in \mathcal{F}$ L is bounded. ■

Remark 4.18 *(counter-example)***.** Counter-example:

$$f(x,n) = \frac{x}{x^2 + n^{-2}}.$$

Then $\forall x \in X:\ \sup_{n\in\mathbb{N}} f(x,n)$ bounded, but $\sup_{n\in\mathbb{N}}\sup_{x\in X} f(x,n)$ not bounded, i.e. $\longrightarrow \infty$. //

Theorem 4.19 *(Banach-Steinhaus theorem)***.** Let X be a Banach space and Y be a normed linear space. Let $(L_n)_{n\in\mathbb{N}} \in (\mathcal{L}(X,Y))^{\mathbb{N}}$ be a sequence of maps. Suppose that for each $x \in X$ the limit $\lim_{n\to\infty} L_n x$ exists. Denote $L\colon X \to Y,\ Lx = \lim_{n\to\infty} L_n x$. Then $L \in \mathcal{L}(X,Y)$, in particular L is continuous. □

Proof. Later. ■

Repitition:

Theorem 4.13 *(Baire category theorem)***.** A complete metric space X cannot be countable union of its nowhere dense sets. □

Theorem 4.17 *(uniform-boundedness principle)***.** Let $\mathcal{F}$ be family of bounded linear maps $X \to Y$, where X is a Banach space, then

$$\left(\forall x \in X:\ \sup_{L\in\mathcal{F}}\|Lx\| < \infty\right) \Leftrightarrow \left(\sup_{L\in\mathcal{F}}\|L\|\right)$$

□

2015-06-23

Remark 4.20. Let $f \in C([0,1])$ and $\varepsilon > 0$.

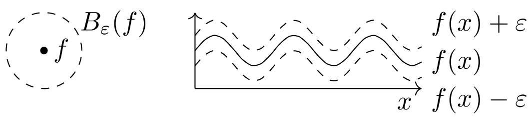


Figure 14: ε-ball around function $f = \varepsilon$ strip following the function f.

//

Lemma 4.21. $C^\infty([0,1])$ is dense in $C([0,1])$. □

Proof. Let $f \in C([0,1])$, then mollifier f_δ is

$$f_\delta(x) := \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\delta} \exp\left(-\frac{(x-y)^2}{2\delta^2}\right) \cdot f(y)\,\mathrm{d}y\,.$$

Graph of $\frac{1}{\sqrt{2\pi}\delta}\exp\left(-\frac{t^2}{2\delta^2}\right)$:

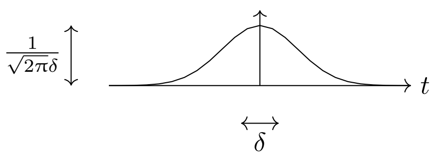


Figure 15: Graph of $\frac{1}{\sqrt{2\pi}\delta}\exp\left(-\frac{t^2}{2\delta^2}\right)$.

Note that

$$\forall \delta > 0 :: f_\delta \in C^\infty([0,1]), \quad f_\delta(x) \xrightarrow[\text{uniformly}]{\delta \to 0} f(x).$$ ■

Theorem 4.22 *(set of somewhere differentiable functions is first category set in $C([0,1])$)*. Let

$$A = \{f \in C([0,1]) \mid \exists x \in [0,1] : f'(x) \text{ exists}\},$$

where

$$f'(x) \text{ exists} \Leftrightarrow \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \text{ exists.}$$

The set $A \subseteq C([0,1])$ is a first category set. In particular $A \neq C([0,1])$. □

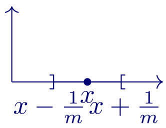
Proof. We express

$$A = \bigcup_{n,m} A_{n,m},$$

where $A_{n,m}$ are closes nowhere dense sets.

$$A_{n,m} = \left\{ f \in C([0,1]) \;\middle|\; \exists x \, \forall y,\ 0 < |x-y| < \tfrac{1}{m} : \Rightarrow \left| \frac{f(x) - f(y)}{x - y} \right| \leq n \right\}.$$

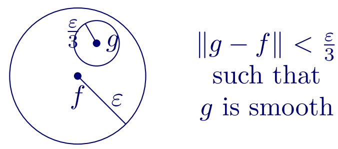
If $f \in A$, then exists x for which $(*)$ exists, then exists n, m such that $f \in A_{n,m}$. It follows that $A = \bigcup_{n,m} A_{n,m}$.



Closed: Let $f_k \in A_{n,m}$ such that $f_k \longrightarrow f \in C([0,1])$. Then there exists points x_k such that for all y satisfying $0 < |x_k - y| < \frac{1}{m}$ it holds that $|\frac{f_k(x_k) - f_k(y)}{x_k - y}| \leq n$. We have a sequence $x_k \in [0,1]$, so there exists a subsequence $x_k \to x \in C([0,1])$. Thne $f_k(x_k) \to f(x)$. Then we have

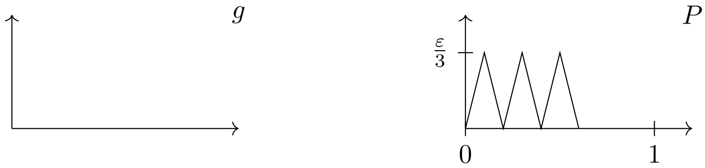
$$\forall y,\ 0 < |x-y| < \tfrac{1}{m} : \left| \frac{f(x) - f(y)}{x - y} \right| = \lim_{k \to \infty} \left| \frac{f_k(x_k) - f_k(y)}{x_k - y} \right| \leq n$$

Nowhere dense: Since $A_{n,m}$ is closed, we need to check that no ball is inside $A_{n,m}$.
Let $f \in A_{n,m}$ and $\varepsilon > 0$, then there exists $h \in B_\varepsilon(f)$ such that $h \notin A_{n,m}$.
For function g:



$$\sup_{\substack{x,y \\ x \neq y}} \left| \frac{g(x) - g(y)}{x - y} \right| < M$$

Claim: The function $g = g + P$ does not belong to $A_{n,m}$.



$|\frac{P(x) - P(y)}{x - y}| \geq M + n + 1$ provided $0 < |x - y| < \frac{3}{\varepsilon(M+n+1)}$

Figure 16: ...

$$\left| \frac{h(x) - h(y)}{x - y} \right| = \left| \frac{g(x) - g(y)}{x - y} + \frac{P(x) - P(y)}{x - y} \right|$$

Then:

$$\inf_{y : 0 < |x-y| < \frac{3}{\varepsilon(M+n+1)}} \left| \frac{h(x) - h(y)}{x - y} \right| \geq M + n + 1 - M = n + 1 > n$$

Therefore:

$$h \notin A_n, m$$

Note:

$$|a + b| \geq ||a| - |b||$$

$$\|P\| < \tfrac{\varepsilon}{3}, \quad \|f - g\| < \tfrac{\varepsilon}{3}, \quad \|f - g - P\| \leq \|f - g\| + \|P\| \leq \tfrac{2\varepsilon}{3}$$ ■

Theorem 4.19 *(Banach-Steinhaus theorem).* Let X be a Banach space and $L_n \in \mathcal{L}(X,Y)$. Suppose that for all $x \in X$ $\lim_{n\to\infty} L_n x$ exists and denote $Lx := \lim_{n\to\infty} L_n x$. Then $L \in \mathcal{L}(X,Y)$. □

Proof. L is linear. L is bounded since $L_n x$ converges for all x. Therefore

$$\forall x \in X: \sup_{n\in\mathbb{N}} \|L_n x\| < \infty \quad \overset{\text{uniform boundedness principle}}{\therefore} \quad \sup_{n\in\mathbb{N}} \|L_n\| < \infty.$$

Then let M such that $\sup_{n\in\mathbb{N}} \|L_n\| < M$, then we have

$$\|Lx\| = \lim_{n\to\infty} \|L_n x\| \leq \sup_{n\in\mathbb{N}} \|L_n x\| \leq \sup_{n\in\mathbb{N}} \|L_n\| \|x\| \leq M\|x\|.$$ ■

Prop. 4.23. Suppose that X is a normed linear space and $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ is weakly converging. Then $(x_n)_{n\in\mathbb{N}}$ is bounded. □

Theorem 4.24. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence ($\sup_{n\in\mathbb{N}} \|x_n\| \leq \infty$) in a *reflexive* Banach space. Then $(x_n)_{n\in\mathbb{N}}$ has a weakly converging subsequence.
Rephrasing: The closed unit ball in a reflexive Banach space is weakly sequentially compact. □

Remark 4.25 *(nets).* Nets is a generalization of sequences, e.g. they fix the difference "compactness ↔ sequential compactness". //

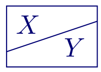
Prop. 4.26 *(closed subspaces of reflexive Banach spaces).* Let X be reflexive Banach space and Y a closed subspace. Then:

(a) Y is reflexive Banach space.

(b) If Y is separable, then Y^* is separable. □

Proof.

(a) Let $\tilde{\varphi} \in Y^*$ and $\varphi \in X^*$. If $\varphi \in X^*$ then $\varphi|_Y \in Y^*$. If $\tilde{\varepsilon} \in Y^{**}$ then $\varepsilon \in X^{**}$, $\varepsilon(\varphi) = \tilde{\varepsilon}(\varphi|_Y)$.



We need to prove that for all $\tilde{\varepsilon} \in Y^{**}$ there exists $y \in Y$ such that $\tilde{\varepsilon}(\tilde{\varphi}) = \tilde{\varphi}(y)$, i.e. $\tilde{\varepsilon} = \tilde{J}_y$. We know that there exists $x \in X$ such that $\varepsilon(\varphi) = \varphi(x)$. Suppose that $x \notin Y$, then there exists $\varphi \in X^*$ such that $\varphi(x) = 1$ and $\varphi|_Y = 0$. Then

$$0 = \tilde{\varepsilon}(\varphi|_Y) = \varepsilon(\varphi) = \varphi(x) = 1,$$

contradicition, hence $x \in Y$. We need to prove that for all $\tilde{\varphi} \in Y^*$ indeed $\tilde{\varepsilon}(\tilde{\varphi}) = \tilde{\varphi}(y)$.

(b) Omitted. ■

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Question: Let $L \in \mathcal{L}(X,Y)$ and suppose $x_n \xrightarrow{\text{w}} x$; does it imply that $Lx_n \xrightarrow{\text{w}} Lx$?
Answer: Yes.
Proof: If $\varphi \in Y^*$.

$$(L'(\varphi))(x_n) = \varphi(L(x_n)) \longrightarrow \varphi(L(x)) = (L'(\varphi))(x), \qquad \|L\| = \|L'\|$$

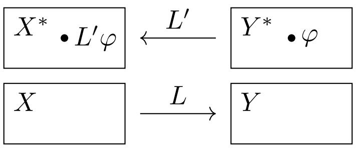
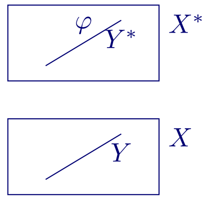


Figure 17: Illustration of the dual of a linear map.

Theorem (later): Suppose that $L \in \mathcal{L}(X,Y)$ is compact. Then $x_n \xrightarrow{\text{w}} x$ implies $Lx_n \longrightarrow Lx$.

Prop. 4.27 *(closed subspaces of reflexive Banach spaces).* Let X be a reflexive Banach space and Y a closed subspace of X. Then Y is a reflexive Banach space.

$$X^* \ni \varphi \longrightarrow \varphi|_Y \in Y^* \qquad \varepsilon(\varphi) := \tilde{\varepsilon}(\varphi|_Y)$$
$$X^{**} \ni \varepsilon \longleftarrow \tilde{\varepsilon} \in Y^{**}$$



□

Proof. We proved that there exists $x \in Y$ such that $\varepsilon(\varphi) = \varphi(x)$. We need to check for all $\tilde{\varphi} \in Y^*$ we have $\tilde{\varepsilon}(\tilde{\varphi}) = \tilde{\varphi}(x)$. By Hahn-Banach there exists $\varphi \in X^*$ such that $\varphi|_Y = \tilde{\varphi}$. Then we have

$$\tilde{\varepsilon}(\tilde{\varphi}) = \varepsilon(\varphi) = \varphi(x) \overset{x\in Y \text{ and } \varphi|_Y = \tilde{\varphi}}{=} \tilde{\varphi}(x).$$ ■

Prop. 4.28 *(dual space of separable reflexive Banach spaces is separable)***.** If X is separable reflexive Banach space, then X^* is separable. □

Proof. Omitted. ■

Theorem 4.29. Let X be reflexive Banach space and $(x_n)_{n=1}^\infty$ a bounded sequence. Then there exists weakly converging subsequence. □

Proof. Let $Y = \overline{\text{span}(\{x_n\}_{n=1}^\infty)}$, then $Y \subseteq X$ and it is a closed linear subspace. We know that Y is reflexive and Y^* is separable (because $Y \ni y \simeq \sum_{n=1}^N \alpha_n x_n$, now choose $\alpha_n \in \mathbb{Q}$). We need to prove that there exists subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(\varphi(x_{n_k}))_{k\in\mathbb{N}}$ converges for all $\varphi \in Y^*$.

- $(\varphi(x_n))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{F}$, $|\varphi(x_n)| \le \|\varphi\|\|x_n\|$.
- We have $\varphi_1, \varphi_2, \ldots \in Y^*$ such that $(\varphi_n)_{n=1}^\infty$ is dense in Y^*.

We have a countable number of sequences $(\varphi_n(x_n))_{n\in\mathbb{N}}$.
Claim: We can find a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ ($y_n = x_{J(n)}$, where $J\colon \mathbb{N} \to \mathbb{N}$ and φ is non-decreasing) such that $(\varphi_k(y_n))_{n\in\mathbb{N}}$ converges for all $k \in \mathbb{N}$.
Diagonal trick:
- Let $(x_n^{(1)})_{n\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $(\varphi_1(x_n^{(1)}))_{n\in\mathbb{N}}$ converges.
- Let $(x_n^{(2)})_{n\in\mathbb{N}}$ be a subsequence of $(x_n^{(1)})_{n\in\mathbb{N}}$ such that $(\varphi_2(x_n^{(2)}))_{n\in\mathbb{N}}$ converges.
- Let $(x_n^{(k)})_{n\in\mathbb{N}}$ be a subsequence such that $(\varphi_1(x_n^{(k)}))_{n\in\mathbb{N}}, \ldots, (\varphi_k(x_n^{(k)}))_{n\in\mathbb{N}}$ converges.
- Put $y_n := x_n^{(n)}$, then $(\varphi_k(y_n))_{n\in\mathbb{N}}$ converges for all k: Fix k, then $(y_n)_{n\in\mathbb{N}}$ for $n \ge k$ is a subsequence of $x_n^{(k)}$.

We got $(y_n)_{n\in\mathbb{N}}$ subsequence of $(x_n)_{n\in\mathbb{N}}$ such that $(\varphi_k(y_n))_{n\in\mathbb{N}}$ converges for all $k \in \mathbb{N}$. Hence for all $\varphi \in Y^*$ each $(\varphi(y_n))_{n\in\mathbb{N}}$ converges. Let ε be given, and find k such that $\|\varphi - \varphi_k\| \le \frac{\varepsilon}{3}$ and N such that $\forall m, n > N$: $|\varphi_k(y_n) - \varphi_k(y_m)| < \frac{\varepsilon}{3}$. Recall that $(x_n)_{n\in\mathbb{N}}$ is bounded, and hence $\|y_n\| \le M$. Then

$$|\varphi(y_n) - \varphi(y_m)| < |\varphi_k(y_n) - \varphi_k(y_m)| + |\varphi(y_n) - \varphi_k(y_n)| + |\varphi(y_m) - \varphi_k(y_m)| < \tfrac{\varepsilon}{3} + 2\tfrac{\varepsilon}{3}M.$$

Let $\varepsilon(\varphi) := \lim_{n\to\infty} \varphi(x_n)$. then (by Banach-Steinhaus theorem) $\varepsilon \in Y^{**}$. By reflexivity $\varepsilon(\varphi) = \varphi(y)$, we claim $y_n \xrightarrow{\text{w}} y$. We know that for all $\varphi \in Y^*$ it holds that $\varphi(y_n) \longrightarrow \varphi(y)$. So $\forall \varphi \in X^*$: $\varphi(y_n) = \varphi|_Y(y_n)$ and hence $\forall \varphi \in X^*$: $\varphi(y_n) \to \varphi(y)$, which is equivalent to $y_n \xrightarrow{\text{w}} y$. ■

Theorem 4.30. Suppose that $L \in \mathcal{L}(X, Y)$ is compact. Then $x_n \xrightarrow{\text{w}} x$ implies $Lx_n \longrightarrow Lx$. □

Proof. We know:

1. Since $x_n \xrightarrow{\text{w}} x$, then $(x_n)_{n\in\mathbb{N}}$ is bounded.
2. Then $(Lx_n)_{n\in\mathbb{N}}$ (as a set) is relatively compact, and also $Lx_n \xrightarrow{\text{w}} Lx$.

Claim: Norm and weak convergence on compact sets conincide. Proof: Suppose that Lx_n does not converge to Lx, then there exists ε and subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\forall k \in \mathbb{N}$: $\|Lx - Lx_{n_k}\| \ge \varepsilon$. $(Lx_{n_k})_{k\in\mathbb{N}}$ has a subsequence $(Lx_{n_k}^{(2)})_{k\in\mathbb{N}}$ such that $Lx_{n_k}^{(2)} \longrightarrow y$ and hence $Lx_{n_k}^{(2)} \xrightarrow{\text{w}} y$. On the other hand $\|y - Lx\| > \varepsilon$, but $Lx_{n_k}^{(2)} \xrightarrow{\text{w}} Lx$, contradition. This proves the claim. Illustration:

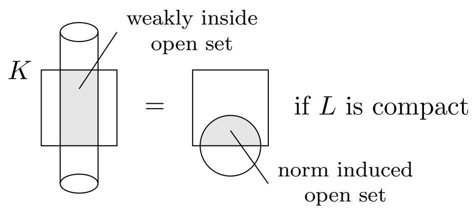


Figure 18: ...

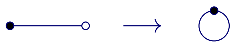
■

Prop. 4.31 *(characterization of weak convergence in compact spaces)***.** Norm and weak convergence on compact sets conincide. □

Proof. See above. ■

Remark 4.32 *(historical remark)***.** Hilbert called operators that map weakly convergent sequences to norm convergent seuqneces totally continuous. Then Riesz introduced compact operators. //

4.4 Open Mapping Theorem and its Corollaries

- Question: Suppose that $L \in \mathcal{L}(X,Y)$ is a bijection; is then L^{-1} bounded?
- Recall bounded ⇔ continuous. Do all continuous bijections have continuous inverse?
- Example: $f\colon [0,2\pi[\to S_1,\ t \mapsto (\cos t, \sin t)$. Illustration: 
- f^{-1} is continuous, if for all open sets U in X $(f^{-1})^{-1}[U] = f[U]$ is open in Y.

Definition 4.33 *(open map)***.** A map f is called open if for each U open also $f[U]$ is open.

Prop. 4.34 *(characterization of open maps)***.** A function is open iff ot maps all neighborhoods of x into neighborhoods of $f(x)$. □

Prop. 4.35. A continuous bijection has continuous inverse of f is open. □

Theorem 4.36. Suppose that X and Y are Banach spaces. Then every $L \in \mathcal{L}(X,Y)$ such that $L[X] = Y$ is open. □

2015-06-30

Repitition: Let X, Y topological spaces and $f\colon X \to Y$ a map.

f is open	$:\Leftrightarrow$	image of open set is open
f is continuous	$:\Leftrightarrow$	preimage of open set is open
f is homeomorphism	$:\Leftrightarrow$	f is continuous bijection with continuous inverse

A neighborhood $V \subseteq X$ of $x \in X$ is a set iff there exists $U \subseteq X$ open such that $x \in U$ and $U \subseteq V$.

Question: Under what conditions a continuous bijection has contiuous inverse.

Prop. 4.37. f is open iff it maps neighborhoods to neighborhoods. □

Proof.

- "⇒": Picture:

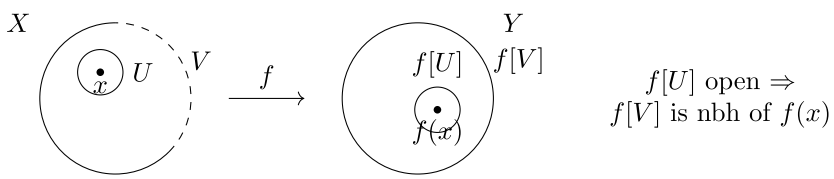


Figure 19: Proof of "f open ⇒ f maps neighborhoods to neighborhoods".

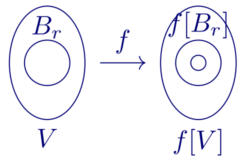
- "⇐": A set is open if it is a neighborhood of all its points. Now assume f maps neighborhoods to neighboorhoods, then for all V open and $x \in V$, it follows that $f[V]$ is neighborhood of $f(x)$, and hence $f[V]$ is open.

■

Theorem 4.38 *(open mapping principle)***.** Let X, Y be Banach spaces and $L \in \mathcal{L}(X, Y)$. Assume that L is surjective, i.e. $L[X] = Y$, then L is open. □

Proof. Steps:

1. Observations: We need to check if L maps neighborhoods to neighborhoods. If V is a neighborhood of x then $-x + V$ is a neighborhood of 0,

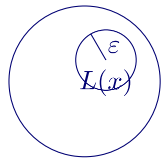


$$L[-x + V] = -L(x) + L[V].$$

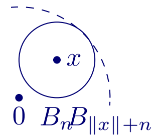
   Each neighborhood V of 0 includes a ball $B_r \subseteq V$ for some $r > 0$. We need to check that B_r is mapped into a neighborhood.

2. To show: There exists ball B_r for some r such that $B_r \subseteq L[B_1]$.

   We have $X = \bigcup_{n \in \mathbb{N}} B_n$, and therefore $Y = L[X] = \bigcup_{n \in \mathbb{N}} L[B_n]$. By Baire category theorem there exists $n \in \mathbb{N}$ such that the interior of $\overline{L[B_n]}$ is non-empty, i.e. there exists $y \in Y$ and $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq \overline{L[B_n]}$. By assumption there exists $x \in X$ such that $y = L(x)$, and hence $B_\varepsilon(L(x)) \subseteq \overline{L[B_n]}$. It follows that:



$$\begin{aligned} B_\varepsilon &= -L(x) + B_\varepsilon(L(x)) \\ L(x) + B_\varepsilon &= B_\varepsilon(L(x)) \subseteq \overline{L[B_n]} \\ B_\varepsilon &\subseteq \overline{L[-x + B_n]} \subseteq \overline{L[B_{n+\|x\|}]} \\ \overline{L[B_1]} &= \tfrac{1}{n+\|x\|}\overline{L[B_{n+\|x\|}]} \quad \therefore \quad B_{\frac{\varepsilon}{n+\|x\|}} \subseteq \overline{L[B_1]} \end{aligned}$$



3. We aim to prove: L maps open sets to open sets.
   We proved: there exists $d > 0$ such that $B_d \subseteq \overline{L[B_1]}$.
   We need to get of of closure. We are going to prove $B_d \subseteq L[B_2]$.
   By approximation:
   - There exists $x_1 \in B_1$ such that $\|L(x_1) - L(x)\| < \frac{d}{2}$. Let me call $y_n = L(x_1) - L(x)$, then $y_1 \in B_{d/2} \subseteq \overline{L[B_{1/2}]}$.
   - There exists $x_2 \in B_{1/2}$ such that $\|L(x_2) - y_1\| < \frac{d}{4}$. Again $y_2 = L(x_2) - y_1$, then $y_2 \in B_{d/4} \subseteq \overline{L[B_{1/4}]}$.
   - Continuing this process we find $x_n \in B_{1/2^{n-1}}$, i.e. $\|x_n\| < \frac{1}{2^{n-1}}$, such that $\|L(x) - L(x_1 + \ldots + x_n)\| < \frac{d}{2^n}$.

   Now I put $x = \sum_{n=1}^\infty x_n = \lim_{N\to\infty} \sum_{n=1}^N x_n$; this limit exists because $\sum_{n=1}^\infty \|x_n\| = \sum_{n=1}^\infty \frac{1}{2^{n-1}} = 2$, and therefore $\|x\| < 2$. Why $\sum_{n=1}^\infty \|x_n\| < \infty \Rightarrow \sum_{n=1}^\infty x_n$ exists? Because X is Banach. In fact

$$X \text{ Banach} \quad \Leftrightarrow \quad \left(\forall (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \sum_{n=1}^\infty \|x_n\| < \infty \Rightarrow \sum_{n=1}^\infty x_n \text{ exists} \right).$$

   For any $y \in B_d$ we found $x \in B_2$ such that $y = L(x)$, i.e. $B_d \subseteq L[B_2]$.

4. We proved $B_{d/2} \subseteq L[B_1]$, therefore L is open.
   Proof of this conclusion: Let V be a neigborhood of $x \in X$. Then there exists a ball $B_\varepsilon(x) \subseteq V$. Then $-x + V$ is a neighborhood of 0 and $B_\varepsilon \subseteq -x + V$. Then $L[B_\varepsilon] \subseteq L[-x + V]$, and therefore $B_{\varepsilon \cdot d/2} \subseteq L[B_\varepsilon] \subseteq L[-x + V]$, hence $B_{\varepsilon \cdot d/2}(L(x)) \subseteq L[V]$. This proves that $L[-x + V]$ is a neighborhood of 0. This also proves that $L[V]$ is a neighborhood of $L(x)$, hence L is open.

5. Further remarks:

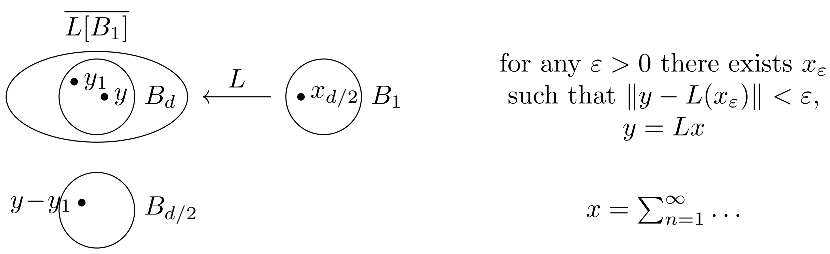


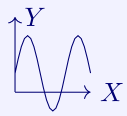
Figure 20: Illustration for the proof of the open mapping principle.

■

Theorem 4.39 *(inverse mapping theorem)***.** Let X, Y be Banach spaces and $L \in \mathcal{L}(X, Y)$ be a bijection. Then $L^{-1} \in \mathcal{L}(Y, X)$. □

Proof. If L is bijection then $L[X] = Y$ and hence L is open. Then open continuous bijection is homeomorphism. ■

Definition 4.40 *(graph of a map)***.**
Let X, Y be normed linear spaces and $L\colon X \to Y$ a map. Then the graph $\Gamma(L)$ of L is defined as $\Gamma(L) = \{(x, y) \in X \times Y \mid y = L(x)\} \subseteq X \times Y$.



Remark 4.41. Recall that $X \times Y$ can be equipped with norm $\|(x, y)\| = \|x\| + \|y\|$. Then $X \times Y$ is normed linear space and if X, Y is Banach, then so is $X \times Y$. //

Theorem 4.42 *(closed graph theorem)***.** Let X, Y be Banach spaces and $L\colon X \to Y$ a linear map. Then the following is equivalent:

(1) L is bounded.

(2) $\Gamma(L)$ is closed. □

Repitition: For $L\colon X \to Y$ graph of L is $\Gamma(L) = \{(x, y) \in X \times Y \mid y = L(x)\}$.

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Theorem 4.42 *(closed graph theorem)***.** Let X, Y be Banach spaces and $L\colon X \to Y$ linear. Then the following is equivalent:

(i) L is bounded.

(ii) $\Gamma(L)$ is closed (as a subspace of $(X \times Y, \|(x, y)\| = \|x\|_X + \|y\|_Y)$). □

Proof.

- "(i) $\Rightarrow$ (ii)": Let L be bounded and $(x_n, L(x_n)) \longrightarrow (x, y)$. We need to check $(x, y) \in \Gamma(L) \;\Leftrightarrow\; y = L(x)$. Indeed, because L is continuous, $x_n \longrightarrow x$ implies $L(x_n) \longrightarrow L(x)$, and hence $L(x) = y$.
- "(ii) $\Rightarrow$ (i)": $X \times Y$ is a Banach space, and by assumption $\Gamma(L)$ is closed, therefore $\Gamma(L)$ is a Banach space.

  Coordinate projections (functions):

  $$\pi_X\colon X \times Y \to X,\ (x, y) \mapsto x$$
  $$\pi_Y\colon X \times Y \to Y,\ (x, y) \mapsto y$$

  For $(x, L(x)) \in \Gamma(L)$ we have $\pi_X(x, L(x)) = x$ and $\pi_Y(x, L(x)) = y$ and

  $$\pi_Y(\pi_X^{-1}(x)) = L(x),$$

  where π_X^{-1} exists as operator $\pi_X^{-1}\colon X \to \Gamma(L)$, because the operator $\pi_X\colon \Gamma(L) \to X$ is a bijection. Then $\pi_Y \circ \pi_X^{-1}\colon X \to Y$.

Claim: π_X, π_Y are bounded maps and π_X^{-1} is a bounded map $X \to \Gamma(L)$. Proof:
π_Y is bounded: $\|\pi_Y(x,y)\| = \|y\| \leq \|x\| + \|y\| \therefore \|\pi_Y\| \leq 1$.
π_X^{-1} bounded as map $X \to \Gamma(L)$: $\pi_X : \Gamma(L) \to X$ is a bijection and $\Gamma(L), X$ are Banach spaces, therefore π_X^{-1} is bounded by the inverse map theorem.

Now consider map

$$\pi_Y \circ \pi_X^{-1} \colon X \to \Gamma(L) \to Y, \ (\pi_Y \circ \pi_X^{-1})(x) = L(x),$$

then $\pi_Y \circ \pi_X^{-1} = L$, and hence L is bounded as composition of two bounded maps.

■

Remark 4.43. unbounded operators $\neq$ not bounded operators //

4.4.1 Application 1: Hellinger-Toeplitz theorem

→ see exercises.

4.4.2 Application 2: Projections on Banach spaces

Definition 4.44 *(kernel, range)*. For a linear operator $L \colon X \to Y$:

$$\begin{aligned} &\text{Kernel:} \quad \ker(P) = \{x \in X \mid L(x) = 0\} &&\subseteq X \\ &\text{Image:} \quad \operatorname{im}(P) = \{y \in Y \mid \exists x \in X: \ y = L(x)\} &&\subseteq Y \end{aligned}$$

Definition 4.45 *(projection)*. Let X be a linear space. A linear operator $P \colon X \to X$ is called *projection* if $P \circ P = P$.

Prop. 4.46. If P is a projection and $x \in X$, then there exists an unique decomposition $x = y + z$ such that $y \in \operatorname{im}(P)$ and $z \in \ker(P)$. □

Proof. Existence:

$$x = \underbrace{P(x)}_{\in \operatorname{im}(P)} + \underbrace{(1-P)(x)}_{\in \ker(P)}$$

We need to check $(P(1-P))(x) = (P - P^2)(x) = (P-P)(x) = 0$. Uniqueness: Suppose $x = y + z$ with $z \in \ker(P)$. Then $P(x) = P(y) + P(z) = P(y) = y$, where the latter inequality follows from $\forall y \in \operatorname{im}(P): \ P(y) = y$, because if $y \in \operatorname{im}(P)$ there exists $x \in X$ such that $y = P(x)$, and $P(y) = P^2(x) = P(x) = y$. ■

Example 4.47. $P_\alpha \colon \mathbb{R}^2 \to \mathbb{R}, \ P_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 0 \end{smallmatrix}\right), \ {P_\alpha}^2 = P_\alpha$.

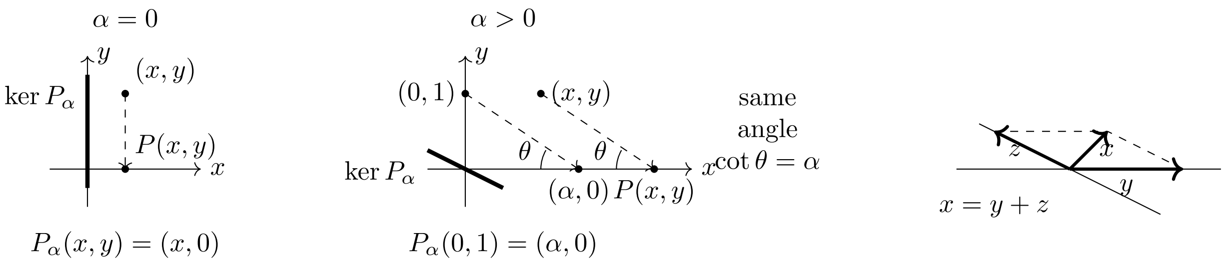

Figure 21: Illustration of the projection $P_\alpha = [[1,\alpha],[0,0]]$.

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Definition 4.48 *(sum of subsets)*. Let X be a lineas space and Y, Z subsets of X. We define $Y + Z = \{x \in X \mid \exists y \in Y, \ z \in Z: \ x = y + z\}$.

Definition 4.49 *(direct sum of linear subspaces)*. Let X be a linear space and Y, Z subspaces of X. Then we write $X = Y \oplus Z$ provided $Y \cap Z = \{0\}$ and $Y + Z = X$. This is equivalent to the existence of a unique decomposition $x = y + z$ where $y \in Y$ and $z \in Z$.

Remark 4.50 *(algebraic ↔ geometric)***.** Given P we define $Y = \operatorname{im}(P)$ and $Z = \ker(P)$. Given $X = Y \oplus Z$, we can define $P\colon X \to X$ given by $P(x) = y$ given $x = y + z$. Claim: P is projection. //

> *Question:* If X is normed linear space, would the decomposition $X = Y \oplus Z$ contiuous? $(x = y + z)$
> *Answer:* This is equivalent to P being bounded.
> *Proof:* $y = P(x)$; $x = P(x) + (1 - P)(x)$; $x_n \to x \;\Rightarrow\; y_n \to y$.

Lemma 4.51. Let X be a normed linear space and L a bounded map on X. Then $\ker(L)$ is closed linear subspace. □

Proof. Let $(x_n)_{n\in\mathbb{N}} \in (\ker(L))^{\mathbb{N}}$ be such that $x_n \overset{n\to\infty}{\longrightarrow} x$. By continuity of L we have $0 = L(x_n) \overset{n\to\infty}{\longrightarrow} L(x)$, therefore $L(x) = 0$, i.e. $x \in \ker(L)$. ■

Theorem 4.52. Let X be a Banach space and Y, Z two subspaces such that $X = Y \oplus Z$. Then the following is equivalent:

(i) Associated projection P is bounded.

(ii) Y, Z are closed. □

Proof.

- "(i) ⇒ (ii)": Put $Y = \operatorname{im}(P)$ and $Z = \ker(P)$. Then Z is closed by the lemma above, and Y is closed because $Y = \ker(1 - P)$.
  Let's proof $Y = \ker(1 - P)$: "$\subseteq$": If $y \in \operatorname{im}(P)$ then $y \in \ker(1 - P)$, because $y = P(x)$ implies $(1 - P)(y) = (1 - P)(P(x)) = (P - P)(x) = 0$. "$\supseteq$": Let $y \in \ker(1 - P)$, then $(1 - P)(y) = 0$, hence $y = P(y)$.

- "(ii) ⇒ (i)": Suppose that Y, Z are closed. We want to show that $P(x) = y$ $(x = y + z)$ is bounded.

$$\Gamma(P) = \{(x, y) \in X \times Y \mid x = y + z\}$$

  1st version of the proof:
  $\Gamma(P)$ closed $\Leftrightarrow$ $x_n = y_n + z_n$ and $(x_n, y_n) \to (x, y)$ then $y \in Y$. In particular $x_n \to x$ and $y_n \to y$, and therefore $z_n \to z$. We conclude $x = y + z$.
  2nd version of the proof:
  $\Gamma(P)$ closed $\Leftrightarrow$ $x_n = y_n + z_n$. If $(x_n, y_n) \to (x, y)$ then $(x, y) \in \Gamma(P)$. From $x_n \to x$ and $y_n \to y$ it follows that $z_n \to z$ such that $x = y + z$. Since Y and Z are closed, $y_n \to y$ implies $y \in Y$ and $z_n \to z$ implies $z \in Z$, together this implies $(x, y) \in \Gamma(P)$. By closed graph theorem, this implies that P is a bounded operator.

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> *Repitition:* Banach space $X = Y \oplus Z$ ↔ P projection with $Y = \ker(P)$, $Z = \operatorname{im}(P)$
> Claim: P bounded $\Leftrightarrow$ Y, Z closed

Example 4.53.

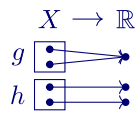
(1) Consider $c = \{(x_n)_{n\in\mathbb{N}} \text{ sequence} \mid \lim_{n\to\infty} x_n \text{ exists}\}$, in particular $c_0 \subseteq c$.
Let Z be a subspace generated by $z = (1, 1, 1, \dots)$. Then $c = c_0 \oplus Z$.

$$\forall x \in c: \; x = \underbrace{x_0}_{\in c_0} + \underbrace{\alpha}_{\in\mathbb{R}} \cdot z$$
$$Px = z(\underbrace{\lim_{n\to\infty} x_n}_{=\alpha})$$

(2) Let (X, Σ, μ) probability space $\mu(X) = 1$. *Random variable* is measureable function $f\colon X \to \mathbb{R}$. For $f \in L^1(X)$ the *expectation value* $\mathbb{E}[F]$ of f is

$$\mathbb{E}[f] = \int_X f \,\mathrm{d}\mu\,.$$

Let $\mathcal{H} = \{f \text{ random variable} \mid \mathbb{E}[f^2] < \infty\}$, then $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space, where $\langle f, g\rangle = \mathbb{E}[f \cdot g]$. Consider a random variable g and subspace $\mathcal{G}$ generated by g,



$$\mathcal{G} = \{h \in \mathcal{H} \mid \exists \text{function } F\colon \mathbb{R} \to \mathbb{R}:\ h = F \circ g \text{ almost surely}\}.$$

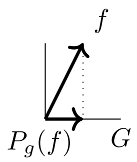
Now orthogonal projection $P_g\colon \mathcal{H} \to \mathcal{H}$ with $\operatorname{im}(P_g) = \mathcal{G}$, i.e. the projection corresponding to $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$. And $P_g(f)$ is *conditional expectation*.
Claim:

$$\forall h \in \mathcal{G}:\ \mathbb{E}[h \cdot \mathbb{E}[f|g]] = \mathbb{E}[h \cdot f] \tag{$*$}$$

Proof:

$$\mathbb{E}[h \cdot \mathbb{E}[f|g]] = \langle h, P_g(f)\rangle = \langle h, P_g(f) + (1 - P_g)(f)\rangle = \langle h, f\rangle = \mathbb{E}[h \cdot f]$$

Comparison to standard definition:
Def.: *Conditional expectation* $\mathbb{E}[f|g]$ is a unique random variable measureable w.r.t. sigma algebra generated by g such that $(*)$ holds.
Claim: P_g is uniquely defined by requirements $(*)$ and $\forall f:\ P_g(f) \in \mathcal{G}$.
Geometric interpretation of random variables:



$(1 - P_g)(f) \in \mathcal{G}^\perp$
is an element
$\operatorname{dist}(\mathcal{G}, f) = \|P_g(f) - f\|$.

Figure 22: Geometric interpretation of random variables

(3) Example of example (2):

T : Temperature of day
A : Amount of icecream sold in a shop

T in °C	34	24	. . .
A in kg	20	10	

$\mathbb{E}[T]$: Average temperature of a day in data
$\mathbb{E}[A]$: Average amount of icecream sold in data
$\mathbb{E}[A|T]$: Average amount sold on days with temperature T

◊

Spectral Theory 5

2015-07-07

5.1 The Spectrum of an Operator

Let X complex Banach space, we consider space $\mathcal{L}(X)$.

Def./Lemma 5.1 *(kernel, image, invertibility)*. For $L \in \mathcal{L}(X)$ we have:

$$\begin{aligned}
&\text{kernel of } L: && \ker(L) &&:= \{x \in X \mid L(x) = 0\} \subseteq X \\
&\text{image of } L: && \operatorname{im}(L) &&:= \{x \in X \mid \exists y \in X:\ x = L(y)\} \subseteq X \\
&\text{invertability of } L: && L \text{ invertible} &&:\Leftrightarrow \exists L^{-1} \in \mathcal{L}(X):\ L^{-1} \circ L = \mathrm{id} = L \circ L^{-1} \\
&\text{inverse map theorem:} && L \text{ invertible} &&\Leftrightarrow \ker(L) = \{0\} \ \wedge\ \operatorname{im}(L) = X
\end{aligned}$$

Definition 5.2 *(spectrum)*. For $L \in \mathcal{L}(X)$ we have:

$$\begin{aligned}
&\textit{resolvent} \text{ of } L: && \varrho(L) &&:= \{\lambda \in \mathbb{C} \mid L - \lambda\mathrm{id} \text{ invertible}\} \subseteq \mathbb{C} \\
&\textit{spectrum} \text{ of } L: && \sigma(L) &&:= \{\lambda \in \mathbb{C} \mid \ker(L - \lambda\mathrm{id}) \neq \{0\} \ \vee\ \operatorname{im}(L - \lambda\mathrm{id}) \neq X\} \subseteq \mathbb{C} \\
&\textit{point spectrum} \text{ of } L: && \sigma_{\mathrm{pt}}(L) &&:= \{\lambda \in \sigma(L) \mid \ker(L - \lambda\mathrm{id}) \neq \{0\}\} = \{\lambda \in \sigma(L) \mid \exists x \neq 0:\ L(x) = \lambda \cdot x\} \subseteq \mathbb{C}
\end{aligned}$$

Theorem 5.3 *(basic properties of the spectrum)*.

(i) $\sigma(L) \cap \varrho(L) = \emptyset$

(ii) $\sigma(L) \cup \varrho(L) = \mathbb{C}$

(iii) $\sigma(L)$ is a compact subset of $\mathbb{C}$ □

Proof.

(i) ✓

(ii) By inverse map theorem for each λ either $\ker(L - \lambda\mathrm{id}) \neq \{0\}$ or $\operatorname{im}(L - \lambda\mathrm{id}) \neq X$ or $L - \lambda\mathrm{id}$ invertible.

(iii) Little bit work.
Claim: $\varrho(L)$ is open subset of $\mathbb{C}$ (this implies $\sigma(L)$ is closed).
Proof: Let $\lambda \in \varrho(L)$, then $(L - \lambda\mathrm{id})^{-1}$ exists by the lemma below. For each $\tilde{\lambda}$

$$\left\|(L - \lambda\mathrm{id} - (L - \tilde{\lambda}\mathrm{id}))\right\| = \left|\lambda - \tilde{\lambda}\right| < \frac{1}{\|(L - \lambda\mathrm{id})^{-1}\|},$$

therefore $L - \tilde{\lambda}\mathrm{id}$ is invertible, in particular $\tilde{\lambda} \in \varrho(L)$, and hence $\varrho(L)$ is open.
We prove (iii) by proving that $\Gamma(L)$ is bounded. ■

Lemma 5.4 *(invertiblity is perserved under small perturburations)*. Let $L \in \mathcal{L}(X)$ be invertible and $S \in \mathcal{L}(X)$ such that $\|S - L\| < \|L^{-1}\|^{-1}$, then S is invertible. □

Proof. We calculate:

$$S = S - L + L = L \circ (L^{-1} \circ (S - L) + 1)$$

We are going to use the geometric series:

$$\forall x \in \mathbb{C}, |x| < 1: \ \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{resp.} \quad \forall x \in \mathbb{C}, |x| < 1: \ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Observations:

1. $\sum_{n=0}^{\infty} x^n$ is absolutely converging ($\|x^n\| \leq \|x\|^n$), therefore $\lim_{N\to\infty} \sum_{n=0}^{N} x^n$ exists.
2. $(1 - x)\sum_{n=0}^{N} x^n = (1 - x) \cdot (1 + x + x^2 + \ldots + x^N) = 1 - x^{N+1} \overset{N\to\infty}{\longrightarrow} 1$.

To finish the proof observe that

$$\left\|L^{-1} \circ (S - L)\right\| \leq \left\|L^{-1}\right\| \cdot \|S - L\| < 1,$$

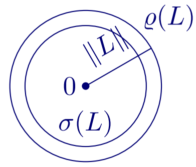
therefore $L^{-1} \circ (S - L) + 1$ is invertible and

$$S^{-1} = (L^{-1} \circ (S - L) + 1)^{-1} \circ L^{-1}.$$ ■

Prop. 5.5. If $|\lambda| > \|L\|$, then $L - \lambda\mathrm{id}$ is invertible, hence $\sigma(L) \subseteq B_{\|L\|}$. □

Proof. $L - \lambda\mathrm{id} = \lambda(\frac{L}{\lambda} - \mathrm{id})$, and since $\|\frac{L}{\lambda}\| < 1$, then

$$(L - \lambda\mathrm{id})^{-1} \overset{\text{Lemma above}}{=} -\frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{L}{\lambda}\right)^n = -\sum_{n=0}^{\infty}\lambda^{-n-1}L^n.$$



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Repitition: X complex Banach space, $L \in \mathcal{L}(X)$.
- Spectrum $\sigma(L) = \{\lambda \in \mathbb{C} \mid \ker(L - \lambda) \neq \{0\} \;\vee\; \mathrm{im}(L - \lambda) \neq \{0\}\}$
- Resolvent $\varrho(L) = \{\lambda \in \mathbb{C} \mid L - \lambda \text{ invertible}\}$
- Claim: $\sigma(L)$ is a compact subset of $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|L\|\}$

5.2 Applications of Spectral Theory

5.2.1 Overview

Overview:

(A) functional calculus

(B) diagonalization

(C) transformation to canonical form

5.2.2 (A) Functional Calculus

Given function $f\colon \mathbb{C} \to \mathbb{C}$, the task is to complete $f(L)$.
Example: For $f(t) = t^2$ we have $f(L) = L^2$.

5.2.3 (B) Diagonalization

Little bit of linear algebra. Consider $X = \mathbb{C}^d$ (finite-dimensional) and $L \in \mathcal{L}(X)$.

Definition 5.6 *(eigenvalues and eigenvectors in finite dimensions)***.** Let $\lambda \in \mathbb{C}$ and $x_\lambda \in X \setminus \{0\}$. If x_λ solves the equation $L(x_\lambda) = \lambda \cdot x_\lambda$, then x_λ *eigenvector* and λ *eigenvalue*.

Remark 5.7. If $\ker(L - \lambda) \neq 0$, then $\lambda \in \sigma_{\mathrm{pt}}(L)$, i.e. λ belongs to the point spectrum of L. //

Prop. 5.8 *(Fredholm alternative)***.** In finite dimensions $\ker(L) \neq 0 \Leftrightarrow \mathrm{im}(L) \neq X$. □

Proof. $L(x) = y$ is solveable iff $\det(L) \neq 0$. ■

Corollary 5.9. $\sigma(L) =$ set of all eigenvalues of L □

5.2.4 (A) Functional Calculus

Theorem 5.10. Assume that L has d linearly independent eigenvectors $(x_n)_{n=1,\ldots,d}$ associated to eigenvalues $(\lambda_n)_{n=1,\ldots,d}$. Then there exists invertible matrix V such that

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} = VLV^{-1}.$$

If $L = L^*$, then $V^{-1} = V^*$ (unitary). In that case:

$$f\left(\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}\right) = \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_d) \end{pmatrix} \quad \text{and} \quad f(VLV^{-1}) = Vf(L)V^{-1}$$

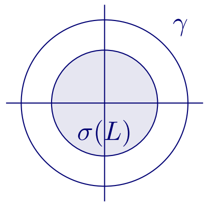
□

Prop. 5.11. f analytic

$$f(L) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-L}\,\mathrm{d}z, \quad \gamma \text{ such that } \sigma(L) \subseteq \operatorname{int}(\gamma)$$

Note: $\frac{1}{z-L} = (z\mathrm{id} - L)^{-1}$. □

Proof.
For diagonal:



$$\frac{1}{2\pi i}\oint_\gamma f(z)\cdot\begin{pmatrix}\frac{1}{z-\lambda_1} & & 0\\ & \ddots & \\ 0 & & \frac{1}{z-\lambda_d}\end{pmatrix}\mathrm{d}z$$

By Cauchy's formula:

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\lambda_1} = f(\lambda_1)$$

■

5.2.5 (C) Transformation to Canonical Form

A quadratic form in $\mathbb{R}^2$: $x = (x_1, x_2)$, $Q(\vec{x}) = 2x_1^2 + 2x_1x_2 + 2x_2^2$, then equation $Q(x) = 1$.

Representation of Q as matrix:

$$Q(x) = \langle x, Lx\rangle, \quad L = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

Diagonalization:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\cdot\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$$

Illustration:

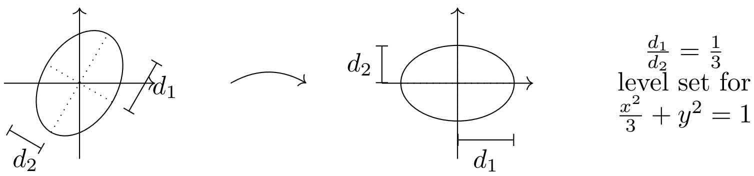


Figure 23: The level sets of quadratic forms on $\mathbb{R}^2$ are ellipses. Diagonalization with unitary matrices align these ellipses with the x- and y-axis

Infinite quadratic form (Hilbert 1906):

$$Q(x) = x_1x_2 + x_2x_3 + x_3x_4 + \dots$$

5.2.6 Overview

Overview of infinite-dimensional functional case:

(A) – Riesz holomorphic functional calculus
– Functional calculus for $L = L^*$

(B) – Diagonalization of maps $L = L^*$
– Spectral theory of compact operators

(C) – Only for hermitian operators

These are the topics of functional analysis II.

5.2.7 General Theory

Definition 5.12 *(dual operator)***.** Recall: X, X^*. For $L \in \mathcal{L}(X)$ define the *dual* $L' \in \mathcal{L}(X^*)$ by

$$(L'(\varphi))(x) = \varphi(L(x)), \;\; \varphi \in X^*, \;\; x \in X.$$

Definition 5.13 *(annihilator)***.** Let M be subspace of X and N subspace of X^*.

$$\begin{aligned} &\textit{annihilator of } M \subseteq X: & M^{\perp} &:= \{\varphi \in X^* \mid \forall x \in M : \varphi(x) = 0 \text{ i.e. } \varphi|_M = 0\} & \subseteq X^* \\ &\textit{annihilator of } N \subseteq X^*: & {}_{\perp}N &:= \{x \in X \mid \forall \varphi \in N : \varphi(x) = 0 \text{ i.e. } \varphi|^N = 0\} & \subseteq X \end{aligned}$$

Lemma 5.14. ${}_{\perp}(M^{\perp}) = \overline{M}$. □

Lemma 5.15. Let $L \in \mathcal{L}(X)$ and denote the dual of L by $L' \in \mathcal{L}(X^*)$. Then:

(i) $(\mathrm{im}(L))^{\perp} = \ker(L')$

(ii) $\ker(L) = {}_{\perp}(\mathrm{im}(L'))$

(iii) $\overline{\mathrm{im}(L)} = {}_{\perp}(\ker(L'))$ □

Proof.

(i) Let $\varphi \in (\mathrm{im}(L))^{\perp}$, this means $\forall x \in X : \;\; \varphi(L(x)) = 0$. Because $0 = \varphi(L(x)) = (L'(\varphi))(x)$ it follows that $L'(\varphi) = 0$, i.e. $\varphi \in \ker(L'$. Let $\varphi \in \ker(L')$, then $0 = (L'(\varphi))(x) = \varphi(L(x))$, and therefore $\forall y \in \mathrm{im}(L) : \; \varphi(y) = 0$, i.e. $\varphi \in (\mathrm{im}(L))^{\perp}$.

(ii) Do it yourself.

(iii) Taking (i) and applying ${}_{\perp}(\cdot)$ implies (iii). ■

5.2.8 (B) Diagonalization

Relevance of L' for diagonalization: We consider $d \times d$ matrix L.
Prop. 5.16. Every eigenvalue of L is also an eigenvalue of L', i.e. $\forall \lambda \in \mathbb{C} : \; \lambda \in \sigma(L) \;\Rightarrow\; \lambda \in \sigma(L')$. □

Proof. Let λ be an eigenvalue of L, then

$$\ker(L - \lambda) \neq 0 \quad \therefore \quad \mathrm{im}(L - \lambda) \neq X \quad \therefore \quad \ker(L' - \lambda) \neq 0,$$

hence λ is an eigenvalue of L'. ■

Prop. 5.17. Let λ be an eigenvalue associated to x_λ. Let further $\tilde{\lambda}$ be an eigenvalue $x_{\tilde{\lambda}}$. If $\lambda \neq \tilde{\lambda}$, then $x_{\tilde{\lambda}} \in \mathrm{im}(L - \lambda)$. □

Proof. If $\lambda \in \sigma(L)$, then $\lambda \in \sigma(L')$, hence $\forall \varphi_\lambda \in X^* : L'(\varphi_\lambda) = \lambda \cdot \varphi_\lambda$. By (iii), for any $x \in \mathrm{im}(L - \lambda)$ we have $\varphi_\lambda(x) = 0$.

$$(L - \lambda)\frac{1}{\tilde{\lambda} - \lambda} x_{\tilde{\lambda}} = \frac{1}{\tilde{\lambda} - \lambda}(\tilde{\lambda} - \lambda) x_{\tilde{\lambda}} x_{\tilde{\lambda}} = x_{\tilde{\lambda}}$$

By $\varphi_\lambda(x) = 0$ it follows that $\forall \tilde{\lambda} \neq \lambda : \;\; \varphi_\lambda(x_{\tilde{\lambda}}) = 0$. ■

Theorem 5.18. Suppose again that L has d distinct eigenvalues with eigenvectors x_λ, then L' has the same eigenvalues to which we can choose φ_λ with $L'(\varphi_\lambda) = \lambda \cdot \varphi_\lambda$ such that

$$L(x) = \sum_{\lambda \in \sigma(L)} \underbrace{\lambda}_{\substack{=:\text{right}\\ \text{eigen-}\\ \text{value}}} \cdot x_\lambda \cdot \underbrace{\varphi_\lambda(x)}_{\substack{=:\text{left}\\ \text{eigen-}\\ \text{value}}}.$$

Why this is diagonalization? It holds that $\varphi_\lambda(x_{\lambda'}) = \delta_{\lambda,\lambda'}$. If $x = \sum_{\lambda \in \sigma(L)} c_\lambda x_\lambda$, then

$$L(x) = \sum_{\lambda \in \sigma(L)} \lambda x_\lambda \varphi_\lambda(x) = \sum_{\lambda \in \sigma(L)} \lambda x_\lambda c_\lambda,$$

where $c_\lambda \in \mathbb{C}$. □

Proof. We know that there exist $\hat{\varphi}_\lambda$ with $\hat{\varphi}_\lambda(x_{\tilde{\varphi}})$ if $\lambda \neq \tilde{\lambda}$. Since $\hat{\varphi}_\lambda$ is nonzero, then we can find $\varphi_\lambda = \#_\lambda \hat{\varphi}_\lambda$ such that $\varphi_\lambda(x_\lambda) = 1$ where $\#_\lambda = \frac{1}{\hat{\varphi}_\lambda(x_\lambda)}$. Therefore we have sets $\{\varphi_\lambda\}_{\lambda\in\sigma(L)}$ (basis of X^*) and $\{x_\lambda\}_{\lambda\in\sigma(L)}$ (basis of X). $\varphi_\lambda(x_{\tilde{\lambda}}) = \delta_{\lambda,\tilde{\lambda}}$. I need to check $L(x_{\tilde{\lambda}}) = \sum_{\lambda\in\sigma(L)} \lambda x_\lambda \varphi_\lambda(x_{\tilde{\lambda}}) = \tilde{\lambda} x_{\tilde{\lambda}}$. ■

5.3 Spectral Theory of Compact Operators

5.3.1 Introduction

2015-07-14

Consider a Hilbert space $L^2(X,\mu) =: \mathcal{H}$. Then for $f \in \mathcal{H}$

$$\|f\|_2 = \int_X |f(x)|^2 \, \mathrm{d}\mu(x) < \infty.$$

Given $\phi \in L^\infty(X,\mu)$, then we define $L_\phi \in \mathcal{L}(\mathcal{H})$ by

$$(L_\phi f)(x) = \phi(x) \cdot f(x).$$

This map has very nice properites:

(a) L_ϕ is bounded:

$$\|L_\phi f\|_2^2 = \int_X |\phi(x)\cdot f(x)|^2 \, \mathrm{d}\mu(x) \leq \|\phi\|_\infty^2 \cdot \|f\|_2^2.$$

(b) The spectrum $\sigma(L_\phi)$ is the essential image of ϕ, i.e.

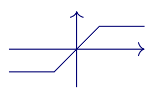
$$\lambda \in \sigma(L_\phi) \quad \Leftrightarrow \quad \forall\varepsilon > 0 : \; \mu(\{x \in X \mid |\phi(x) - \lambda| > \varepsilon\}) > 0.$$

Why? $z \in \varrho(L_\phi)$ iff $(L_\phi - z\mathrm{id})^{-1}$ exists. If $(L_\phi - z\mathrm{id})^{-1}$, then $L_{(\phi-z)^{-1}}(L_\phi - z\mathrm{id})(f) = (\phi - z)^{-1}(\phi - z)f = f$, and $z \in \varrho(L_\phi)$ iff $\frac{1}{\phi - z} \in L^\infty(X,\mu)$.

(c) λ is in the point specturm $\sigma_{\mathrm{pt}}(L_\phi)$ if $\mu(\{x \in X \mid \phi(x) = \lambda\}) > 0$:

$$\begin{aligned} \lambda \in \sigma_{\mathrm{pt}}(L_\phi) \quad &\Leftrightarrow \quad \exists f_\lambda \in L^2(X,\mu)\setminus\{0\} : \; \lambda \cdot f_\lambda(x) = (\lambda \cdot f_\lambda)(x) = (L_\phi f_\lambda)(x) = \phi(x)\cdot f_\lambda(x) \\ &\Rightarrow \quad \exists f_\lambda \in L^2(X,\mu)\setminus\{0\} : \; f_\lambda \text{ is supported on } \{x \in X \mid \phi(x) = \lambda\} \end{aligned}$$

Example: $X = \mathbb{R}$, $\mu = \lambda$.
Consider $\phi(x) = \max\{-a, \min\{x, +a\}\}$.
Then $\sigma(L_\phi) = [-a, +a]$ and $\sigma_{\mathrm{pt}}(L_\phi) = \{-a, +a\}$.



(d) If $\phi = \overline{\phi}$, then $L_\phi = {L_\phi}^*$, i.e. L_ϕ is hermitian:

$$\begin{aligned} \langle f, L_\phi g\rangle &= \int_X \overline{f} \cdot L_\phi g \, \mathrm{d}\mu = \int_X \overline{f(x)} \cdot \phi(x) \cdot g(x) \, \mathrm{d}\mu(x) \\ &= \int_X \overline{\phi(x)\cdot f(x)} \cdot g(x) \, \mathrm{d}\mu(x) = \int_X \overline{L_\phi f} \cdot g \, \mathrm{d}\mu = \langle L_\phi f, g\rangle = \langle f, L_\phi^* g\rangle \end{aligned}$$

(e) Given $F\colon \mathbb{C} \to \mathbb{C}$ bounded and continuous, then

$$(F(L_\phi)f)(x) := F(\phi(x))f(x) \quad \Leftrightarrow \quad F(L_\phi) = L_{F(\phi)}.$$

Check for $F = x^n$: $L_\phi^n f = L_\phi \cdots L_\phi f = \phi^n f = L_{\phi^n} f$.

Theorem 5.19 *(spectral theorem for hermitian operators)***.** Let $\mathcal{H}$ be a Hilbert space and $H \in \mathcal{L}(\mathcal{H})$ with $H = H^*$, i.e. H hermitian. Then there exists measure space (X,Σ,μ) and $\phi \in L^\infty(X)$ and a unitary map $U\colon \mathcal{H} \to L^2(X,\Sigma,\mu)$ such that

$$H = U^* \circ L_\phi \circ U.$$

□

Why do we want to compute functions of operators?

Example 5.20 *(linear ordinary differential equation)***.** Given ODE $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = L(x(t))$ where $x(t) \in X$ and $L \in \mathcal{L}(X)$. The solution of this equation with initial condition $x(0)$ is

$$x(t) = \exp(Lt) \cdot x(0)$$

because

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = L(\exp(Lt) \cdot x(0)), \quad \exp(Lt) = \sum_{n=0}^{\infty} \frac{L^n t^n}{n!}. \qquad \lozenge$$

Example 5.21 *(discrete time)***.** Let $k \in C([0,1]^2)$ and consider map

$$K\colon C([0,1]) \to C([0,1]),\ (Kf)(x) := \int_0^1 k(x,y) \cdot f(y)\,\mathrm{d}y \quad \text{(Fredholm operator)}.$$

Assume that $\forall x: \int_0^1 k(x,y)\,\mathrm{d}y = 1$ and that $\forall x,y: \ k(x,y) \geq 0$. If $p(x)$ is probability density on $[0,1]$, then

$$\int_0^1 k(x,y)p(y)\,\mathrm{d}y$$

is a density (stochastic map). When we apply K again and again on f, then we get a Markov stochastic process in discrete time,

$$p_{n+1} = Kp_n.$$

Solution is $p_n = K^n p_0$. What happens if $n \to \infty$? $\lozenge$

Prop. 5.22 *(a criterion for quasi-nilpotence)***.** If $\sigma(K)$ is strictly bounded in B_1,

$$\forall \lambda \in \mathbb{C}: \ \lambda \in \sigma(K) \ \Rightarrow \ |\lambda| < 1,$$

then

$$K^n \overset{n\to\infty}{\longrightarrow} 0.$$

□

Proof. This follows from Gelfand formula. ■

5.3.2 Spectral Theory of Compact Operators

Example 5.23 *(sounds from instruments)***.** Any sound from instruments can be described with that. For example

$$\varDelta u_\lambda = \lambda u_\lambda, \qquad \varDelta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$$

for $u \in L^2(\varOmega)$, where $\varOmega \subseteq \mathbb{R}^2$ is the shape of the drum. However, $L\colon u \mapsto \varDelta u$ is not bounded (not everywhere defined), in particular non-compact operator. Luckily, $(L - z\mathrm{id})^{-1}$ is compact provided $z \in \varrho(L)$. A map R_z that maps g to a solution of $\varDelta z - zu = g$ is compact for $z \notin \mathbb{R}$. $\lozenge$

Definition 5.24 *(bounded from below)***.** A map $L\colon X \to Y$ between Banach spaces X, Y is called *bounded from below* if

$$\exists C > 0\ \forall x \in X: \ \|Lx\| \geq C^{-1}\|x\|.$$

Lemma 5.25 *(image of bounded-from-below operator is closed)***.** If $L \in \mathcal{L}(X,Y)$ (between Banach spaces X, Y) is bounded from below, then $\mathrm{im}(L)$ is closed. □

Proof. Let $(y_n)_{n\in\mathbb{N}} \in (\mathrm{im}(L))^{\mathbb{N}}$ such that $y_n \overset{n\to\infty}{\longrightarrow} y$. To show $y \in \mathrm{im}(L)$. Because $y_n \in \mathrm{im}(L)$ we have $\exists x_n \in X$: $y_n = Lx_n$, and

$$\|x_n - x_m\| \leq C\|y_n - y_m\| \longrightarrow 0,$$

i.e. $(x_n)_{n\in\mathbb{N}}$ is cauchy, hence $x_n \overset{n\to\infty}{\longrightarrow} x$. It follows that $Lx_n \overset{n\to\infty}{\longrightarrow} Lx$ and thus $y \in \mathrm{im}(L)$. ■

Lemma 5.26 *(image of disturbed bounded-from-below compact operator is closed)***.** Let K be a compact operator on a Banach space X and $\lambda \neq 0$. Then $\operatorname{im}(L - \lambda \mathrm{id})$ is closed. □

Proof. Generally, if $f \in \ker(L - \lambda \mathrm{id})$ with $\|f\| \neq 0$, then $\|(K - \lambda \mathrm{id})f\| = 0$. So $K - \lambda \mathrm{id}$ cannot be bounded from below.

So we need a side step: Decompose $X = \ker(K - \lambda \mathrm{id}) \oplus Y$, where $\ker(K - \lambda \mathrm{id})$ is closed. But $\ker(K - \lambda \mathrm{id})$ being closed subspace is not enough for X to be decomposable. We further need:
Claim: $\ker(K - \lambda \mathrm{id})$ is finite-dimensional.
Proof of claim: $K|_{\ker(K-\lambda \mathrm{id})} = \lambda \mathrm{id}$. If $\ker(K - \lambda \mathrm{id})$ is infinite-dimensional, then id is not compact, contradicition.

Step 2:
Claim: $(K - \lambda \mathrm{id})[Y] = \operatorname{im}(K - \lambda \mathrm{id})$.
Proof of claim: For each $x \in X$ we have $x = z + y$ where $z \in \ker(K - \lambda \mathrm{id})$, and therefore $(K - \lambda \mathrm{id})x = (K - \lambda \mathrm{id})y$ on Y.

$(K - \lambda \mathrm{id})$ is bounded from below. ■

5.3.3 Fredholm alternative

2015-07-17

Theorem 5.27 *(Fredholm alternative)***.** Let K be a compact map on a Banach space X and $\lambda \neq 0$. Then

$$\ker(K - \lambda \mathrm{id}) = 0 \;\Leftrightarrow\; \operatorname{im}(K - \lambda \mathrm{id}) = X.$$ □

Remark 5.28 *(equivalent formulation of the Fredholm alternative)***.** Equation for x with y given: $Kx - \lambda x = y$. Either it has unique solution for all y, or it has a nontrivial solution with $y = 0$. //

Example 5.29 *(nilpotence and $\ker(L) + \operatorname{im}(L) = X$ in finite dimensions)***.** Examples:

Matrix	im	ker	dim ker + dim im	ker ⊕ im
$L_1 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	X	0	$2+0=2$	X
$L_2 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	x-line	y-line	$1+1=2$	X
$L_3 = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	0	X	$0+2=2$	X
$L_4 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$	x-line	x-line	$1+1=2$	x-line
$L_2^{(\alpha)} = \left(\begin{smallmatrix}1&\alpha\\0&0\end{smallmatrix}\right)$				

Table 1: Images and kernels of some linear maps in finite dimensions.

The obstruction for $\ker L + \operatorname{im} L \neq X$ is nilpotence.

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Example in $\mathbb{R}^3$:

$$\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}^3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$ ◊

Example 5.30 *(quasi-nilpotence)***.** Right shift:

$$R\colon \ell^2 \to \ell^2 \text{ defined by } (Rx)_n = x_{n-1},\ (Rx)_1 = 0 \text{ i.e. } (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$$

Then:

$$\begin{aligned} \ker(R) = 0, &\quad \operatorname{im}(R) = \{x \mid x_1 = 0\} \\ \ker(R^n) = 0, &\quad \operatorname{im}(R^n) = \{x \mid x_1 = \ldots = x_n = 0\} \end{aligned}$$ ◊

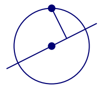
Theorem 5.31 *(Schauder theorem)***.** K is compact iff K' is compact. □

Proof of theorem 5.27.

- "$\ker(K - \lambda \mathrm{id}) = 0 \;\Rightarrow\; \mathrm{im}(K - \lambda \mathrm{id}) = X$": Define $M_n := \mathrm{im}((K - \lambda \mathrm{id})^n)$. Then

$$X = M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n,$$

we will prove that $\exists n \in \mathbb{N} : \; M_n = M_{n+1}$.



First, for construction suppose that M_{n+1} is a proper subspace M_n. [If $e_1 = (1, 0, 0, \ldots)$, then $R^n(e_1) = e_n$.] Reisz lemma: If $U \subseteq X$ is a proper subspace, then there exists $x \in X$ with $\|x\| = 1$ such that $\mathrm{dist}(x, U) > \frac{1}{2}$. By virtue of the Riesz Lemma I can pick $x_n \in M_n$ with $\|x_n\| \in M_n$ such that $\mathrm{dist}(x_n, M_{n+1})$ ($M_{n+1} \subseteq M_n$). Claim: $(Kx_n)_{n\in\mathbb{N}}$ is not cauchy (none of its subsequences). Proof of claim: For $m > n$:

$$\begin{aligned} \|Kx_n - Kx_m\| &= \|(K-\lambda)x_n - (K-\lambda)x_m - \lambda x_m + \lambda x_n\| \\ &= \|y + \lambda x_n\| \\ &= \lambda \left\| \tfrac{1}{\lambda} y + x_n \right\| \\ &> \tfrac{1}{2} \end{aligned}$$

Contradiction to K compact. We conclude $\exists n \in \mathbb{N} : \; M_{n+1} = M_n$.

Claim: $M_{n+1} = M_n \;\Rightarrow\; M_n = M_{n-1}$. Proof of claim: Let $x \in M_{n-1}$. Then:

$x \in M_{n-1} \quad \therefore \quad (K-\lambda)x \in M_n = M_{n+1} = \mathrm{im}(K-\lambda \mathrm{id})^{n+1} \quad \therefore \quad (K-\lambda)x = (K-\lambda)^{n+1} z \quad \therefore \quad x = (K-\lambda)^n z \quad \therefore \quad x \in$

It follows that $M_n \subseteq M_{n-1}$ and hence $M_n = M_{n-1}$.

By induction $\mathrm{im}(K - \lambda \mathrm{id}) = M_1 = M_0 = X$.

- "$\mathrm{im}(K - \lambda \mathrm{id}) = X \;\Rightarrow\; \ker(K - \lambda \mathrm{id}) = 0$": Assume $\mathrm{im}(K - \lambda \mathrm{id}) = X$. By Schauder theorem $\ker(K' - \lambda \mathrm{id}) = 0$. By part 1 $\mathrm{im}(K' - \lambda \mathrm{id}) = X$. It follows that $\ker(K - \lambda \mathrm{id}))0$.

■

Example 5.32 *(Fredholm equation)*.

Fredholm equation of first type: $\displaystyle\int_0^1 K(x,y) \cdot f(y) \,\mathrm{d}y = g(x)$ where g is given

Fredholm equation of second type: $\displaystyle\int_0^1 K(x,y) \cdot f(y) - f(x) \,\mathrm{d}y = g(x)$

K compact, $Kf = g$, $(K - 0)f = g$, $(\lambda - \frac{1}{\lambda})f = g$, if $\frac{1}{\lambda} \in \sigma(K)$ then for each g exists unique solution f. ◊

List of Symbols

Remark by the typesetter: This section is written by the typesetter of the script, and is not part of the lecture itself.

Sequence spaces ($\forall_{\text{cf}}$ = for all except finitely many)

$$\forall p, r \in [1, \infty] : \ p < r \ \Rightarrow \ \ell^p \subsetneq \ell^r \qquad ; \qquad \forall p \in \,]1, \infty[\, : \ \{0\} \subsetneq c_{00} \subsetneq \ell^1 \subsetneq \ell^p \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty = \mathbb{F}_{\text{b}}^{\mathbb{N}}$$

Table 2: Hierarchy of some sequences spaces.

symbol	definition	scalar prod. space	re-flex-ive	com-plete	weak-ly seq. compl.	sepa-rable	isometric isomorphy	comment
$(\mathbb{F}_{\text{b}}^{\mathbb{N}}, \|\cdot\|_\infty)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid x \text{ bounded}\}$	×	×	✓	×	×		
$(c, \|\cdot\|_\infty)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid x \text{ convergent}\}$	×	×	✓	×	✓	$c^* \cong \ell^1$	c closed in $\mathbb{F}_{\text{b}}^{\mathbb{N}}$
$(c_0, \|\cdot\|_\infty)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid x_n \overset{n\to\infty}{\longrightarrow} 0\}$	×	×	✓	×	✓	$(c_0)^* \cong \ell^1$	c_0 closed in $\mathbb{F}_{\text{b}}^{\mathbb{N}}$
$(c_{00}, \|\cdot\|_\infty)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid \forall_{\text{cf}} n \in \mathbb{N} : x_n = 0\}$	×		×		✓		c_{00} dense in c_0, ℓ^2
$(\ell^1, \|\cdot\|_1)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_1 < \infty\}$	×	×	✓	✓	✓	$(\ell_1)^* \cong \ell^\infty$	
$(\ell^2, \|\cdot\|_2)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_2 < \infty\}$	✓	✓	✓	✓	✓	$(\ell_2)^* \cong \ell^2$	
$(\ell^p, \|\cdot\|_p)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_p < \infty\}$	×	✓	✓	✓	✓	$(\ell^p)^* \cong \ell^q$	
$\ell^\infty, \|\cdot\|_\infty)$	$\{x \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_\infty < \infty\}$	×	×	✓	×	×		

Table 3: Some sequence spaces and their properties. Here $p, q \in \,]1, \infty[\, \setminus \{2\}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Function spaces Let X be a set, I an arbitary (index) set, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

X^I	$= \{f : I \to X \ X\text{-valued function on } I\} = \{(x_i)_{i \in I} \ X\text{-valued family over } I\}$	
X_{b}^I	$= \{f \in X^I \mid f \text{ bounded}\}$	X metric space
$X^{(I)}$	$= \{(x_i)_{i \in I} \in X^I \mid \forall_{\text{cf}} i \in I : x_i = 0\}$	
$C(X)$	$= \{f : X \to \mathbb{F} \mid f \text{ continuous}\}$	X topological space

Table 4: Some function spaces.

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