LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Functional Analysis
Exercise Sheet 12

- *Do not hand in this sheet. There will be no correction!*
- *Try to solve it within 2.5 hours.*

Exercise 1 (10 points). *Show that*

$$\phi(f) := \int_{-1}^{1} x\, f(x)\, dx \ , \qquad f \in L^2([-1,1])\,,$$

defines a bounded linear functional $\phi : L^2([-1,1]) \to \mathbb{C}$ and compute its norm.

Exercise 2 (10 Points). *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a linear space X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are complete. Prove that the norms are equivalent if there exists a constant $C > 0$ such that*

$$\|x\|_2 \ \leqslant\ C\,\|x\|_1 \qquad \forall x \in X\,.$$

Exercise 3 (3+5+2+10 Points). *For $k \in C([0,1]^2)$ and $f \in L^1([0,1])$ let*

$$\big(Tf\big)(x) := \int_0^1 k(x,y)\, f(y)\, dy \qquad \forall x \in [0,1].$$

(i) *Verify that $(Tf)(x)$ is well-defined for every $x \in [0,1]$ and every $f \in L^1([0,1])$.*

(ii) *Show that $Tf \in C([0,1])$ for every $f \in L^1([0,1])$.*

(iii) *Prove that $T : (L^1([0,1]), \|\cdot\|_1) \to (C([0,1]), \|\cdot\|_\infty)$ is bounded.*

(iv) *Prove that T is compact.*

Exercise 4 (20 Points). *Let X be a normed space over the field $\mathbb{K}$ and let $(x_n)_{n\in\mathbb{N}}$ be a weak Cauchy sequence in X, which means that $(l(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ for each $l \in X^*$. Prove that $(x_n)_{n\in\mathbb{N}}$ is bounded, i.e.*

$$\sup_{n\in\mathbb{N}} \|x_n\| < \infty\,.$$

Exercise 5 (20 Points). *Let X be a normed space over the field $\mathbb{K}$, let $n \in \mathbb{N}$, let $x_1, \ldots, x_n \in X$ be linearly independent and let $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Show that there exists $l \in X^*$ such that*

$$l(x_j) = \alpha_j \qquad \forall\, j \in \mathbb{N} \text{ with } 1 \leqslant j \leqslant n.$$

Exercise 6 (13+7 Points). *Let X be a normed space.*

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X that converges in the weak topology. Prove that in this case, $(x_n)_{n\in\mathbb{N}}$ converges also in norm.

[Hint: First show that $(f(x_n))_{n\in\mathbb{N}}$ converges uniformly in $f \in \{g \in X^* : \|g\|_* = 1\}$.]

(ii) Conclude: If the closed unit ball $B := \{x \in X : \|x\| \leqslant 1\}$ in X is weakly sequentially compact, then X is complete.

For general informations please visit http://www.math.lmu.de/~gottwald/15FA/