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Summer term 2015
June 22, 2015

Functional Analysis
Exercise Sheet 10

Weak topology and the Uniform Boundedness Principle

- *First version deadline:* **June 29** (13:30). *Final hand in deadline:* **July 13** (13:30)

Exercise 1 (5 points). *Consider the sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R})$ given by $f_n := \chi_{[n,n+1]}$. Decide whether $(f_n)_n$ converges*

(*i*) *almost everywhere,*

(*ii*) *weakly,*

(*iii*) *in norm.*

Exercise 2 (10 points). *Let X be an infinite-dimensional normed linear space. Prove:*

(*i*) *For $x, y \in X$ with $x \neq y$ there exists $\lambda \in X^*$ such that $\|\lambda\| = 1$ and $\lambda(x) \neq \lambda(y)$.*

(*ii*) *A norm-closed convex subset $B \subset X$ is also weakly closed.* [*Hint: You may use the following version of the Hahn-Banach Separation Theorem without proof: If $A, B \subset X$ are disjoint, convex, and non-empty such that A is compact and B is closed, then there exists $\lambda \in X^*$ and $\alpha \in \mathbb{R}$ such that $Re(\lambda(a)) < \alpha < Re(\lambda(b))$ for all $a \in A$ and $b \in B$.*]

(*iii*) *The weak closure of the unit sphere $S \subset X$ is the closed unit ball in X.*

Exercise 3 (5 points). *A function $f : X \to \mathbb{R}$ on a topological space (X, τ) is called lower semicontinuous if the set $\{x \in X : f(x) \leqslant a\}$ is closed for all $a \in \mathbb{R}$. f is called sequentially lower semicontinuous if $x_n \xrightarrow{\tau} x$ implies $f(x) \leqslant \liminf_n f(x_n)$.*

Let $(X, \|\cdot\|)$ be a normed linear space. Decide whether $\|\cdot\|$ is

(*i*) *weakly sequentially continuous,*

(*ii*) *weakly sequentially lower semicontinuous,*

(*iii*) *weakly lower semicontinuous,*

and prove your statements.

Exercise 4 (5 points)**.** *Show that a weakly convergent sequence in a normed linear space is bounded.*

In the following exercise you are asked to prove that for bilinear functions on Banach spaces, continuity in each variable (for the other variable fixed) implies joint continuity. To appreciate the statement try to find an example of a function of two real variables that is continuous in each variable but not jointly continuous.

Exercise 5 (5 points)**.** *Let X and Y be Banach spaces, and let $\tau : X \times Y \to \mathbb{C}$ be such that for each $x \in X$ and $y \in Y$ we have $\tau(x, \cdot) \in Y^*$ and $\tau(\cdot, y) \in X^*$. Prove that if $x_n \to x$ and $y_n \to y$ then $\tau(x_n, y_n) \to \tau(x, y)$ as $n \to \infty$.*

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