LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis
Exercise Sheet 7

Bounded linear maps and Hahn-Banach theorem

- *First version deadline:* **June 8** (13:30). *Final hand in deadline:* **June 22** (13:30)

Exercise 1 (5 points)**.** *Decide for which combinations of* $X, Y \in \{C([-1,1]), L^2([-1,1])\}$, *the prescription*

$$T : X \to Y, \quad Tf(x) := f(x^2)$$

defines a bounded linear map $T \in \mathcal{L}(X, Y)$, *and compute* $\|T\|_{X \to Y}$ *when it does.*

Many special polynomials are the result of an orthogonalization process in a given Hilbert space. Here is an example:

Exercise 2 (5 points)**.** *Do the Gram-Schmidt orthogonalization process of the monomials* $1, x, x^2, x^3$ *in* $L^2[(-1,1)]$ *to obtain the first four Legendre polynomials.*

Note: Up to normalization, the orthogonalization of $1, x, \ldots, x^n$ *yields the* n*th Legendre Polynomial given by*

$$P_n(x) := \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} [(x^2 - 1)^n].$$

Although we said in the lecture that objects in $(\ell^\infty)^*$ do not possess an explicit description, we can still study them. Elements of $(\ell^\infty)^*$ that are not represented by an element in ℓ^1 are sometimes denoted by LIM.

Exercise 3 (5 points)**.** *In the following let* $\ell^\infty := \ell^\infty(\mathbb{N}, \mathbb{R})$ *(i.e. the* $\mathbb{R}$*-vector space of real-valued bounded sequences).*

(*a*) *Prove that there exists a function* $\mathrm{LIM} : \ell^\infty \to \mathbb{R}$ *such that*

(*i*) $\mathrm{LIM}(z + \alpha w) = \mathrm{LIM}(z) + \alpha \mathrm{LIM}(w)$ *for all* $z, w \in \ell^\infty$, $\alpha \in \mathbb{R}$,

(*ii*) $\liminf_{n\to\infty} z_n \le \mathrm{LIM}(z) \le \limsup_{n\to\infty} z_n$ *for all* $z \in \ell^\infty$,

and furthermore that for any $z \in \ell^\infty$ *for which* $\lim_{n\to\infty} z_n$ *exists we have*

(*iii*) $\mathrm{LIM}(z) = \lim_{n\to\infty} z_n$.

(*b*) *What are the possible values of* $\mathrm{LIM}(x)$ *for* $x = (x_n)_{n\in\mathbb{N}}$, *where* $x_n := (-1)^n$*?*

(*c*) *Find the set* $\{(\mathrm{LIM}(x), \mathrm{LIM}(y)) \,|\, \mathrm{LIM}$ *satisfies (i)-(iii) in (a)*$\} \subset \mathbb{R}^2$ *for* x *as in (b) and* $y = (0, 1, 0, 1, 0, \ldots)$.

We remark, that a LIM that is invariant under left-shift, $\mathrm{LIM}(x_1, x_2, \dots) = \mathrm{LIM}(x_2, \dots)$, is called a Banach limit.

The Hahn-Banach theorem has many algebraic and geometric versions and several closely related consequences. Here we show some of its corollaries:

Exercise 4 (5 points)**.** *Let X be a normed linear space, let $Y \subset X$ be a subspace, and let $f \in Y^*$. Prove that there exists $F \in X^*$ such that $F|_Y = f$ and $\|F\|_{X^*} = \|f\|_{Y^*}$.*

Exercise 5 (5 points)**.** *Let X be a normed linear space and $x_0 \in X$.*

(i) *Let Y be a proper subspace of X such that $x_0 \notin Y$ and $d := \mathrm{dist}(x_0, Y) > 0$. Prove that there exists $f \in X^*$ with $f|_Y = 0$, $f(x_0) = d$, and $\|f\|_{X^*} = 1$.*

(ii) *Let V be a closed proper subspace of X such that $x_0 \notin V$. Prove that there exists $f \in X^*$ such that $f|_V = 0$ and $f(x_0) \neq 0$.*

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