Prof. M. Fraas, PhD
A. Groh, S. Gottwald

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Functional Analysis
Exercise Sheet 5

Overview of Banach spaces

- *First version deadline:* **May 22** (12:15). *Final hand in deadline:* **June 8** (13:30)

As was already stressed many times, a Banach space is a basic unit of analysis for FA. In the lecture we already covered basic classification of Banach spaces and this weeks exercise sheet is mainly about an overview of the most common sequence spaces that we encounter in FA.

Consider the following normed spaces:

(i) $(c, \|\cdot\|_\infty)$, *the space of sequences* $\{x_n\}_{n=0}^\infty \subset \mathbb{C}$ *such that* $\lim_{n\to\infty} x_n$ *exists.*

(ii) $(c_0, \|\cdot\|_\infty)$, *the subspace of* c *of sequences with zero limit.*

(iii) $(\ell^p, \|\cdot\|_p)$ *for* $1 \le p < \infty$, *the space of sequences* $\{x_n\}_{n=0}^\infty \subset \mathbb{C}$ *for which* $\sum_{n=0}^\infty |x_n|^p$ *is finite, equipped with* $\|x\|_p := (\sum_{n=0}^\infty |x_n|^p)^{1/p}$.

(iv) $(l^\infty, \|\cdot\|_\infty)$, *the space of bounded sequences* $\{x_n\}_{n=0}^\infty \subset \mathbb{C}$.

Exercise 1 (5 points)**.** *Which of the above spaces are complete? Prove your statements.*

Exercise 2 (5 points)**.** *Which of the above spaces are separable? Prove your statements.*

Exercise 3 (5 points)**.** *Which of the above spaces are inner product spaces? Prove your statements.*

Exercise 4 (5 points)**.** *Let* $\mathcal{H}$ *be a Hilbert space, and let* $\{\phi_n\}_{n=1}^\infty \subset \mathcal{H}$ *be such that* $\|\phi_n\| = 1$ *for all* $n \in \mathbb{N}$ *and*

$$\|f\|^2 = \sum_{n=1}^\infty |(\phi_n, f)|^2$$

for all $f \in \mathcal{H}$. *Show that* $\{\phi_n\}_{n=1}^\infty$ *is an orthonormal basis.*

Exercise 5 (5 points)**.** *A subset* B *of a linear space* X *is called Hamel basis, if* B *is linearly independent*[1] *and every* $x \in X$ *can be written as a finite linear combination of elements from* B. *By using Zorn's Lemma prove that every vector space* $X \neq \{0\}$ *has a Hamel basis.*

For general informations please visit http://www.math.lmu.de/~gottwald/15FA/

[1] For a general linear space X, $B \subset X$ is called linearly independent if all non-empty finite subsets $F \subset B$ are linearly independent.