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Functional Analysis
Exercise Sheet 4

Hilbert Spaces

- *First version deadline:* **May 18** (13:30). *Final hand in deadline:* **June 1** (13:30)

Exercise 1 (5 points). *Here are several statements about the connection of an inner product and the norm it generates.*

(i) *Prove that* $||x|| := \sqrt{(x,x)}$ *is indeed a norm, if* $(\cdot,\cdot)$ *is an inner product.*

(ii) *Show that the inner product in a complex inner product space* V *can be reconstructed from the induced norm by means of the polarization identity*

$$(x,\, y) = \frac{1}{4}\left\{(||x+y||^2 - ||x-y||^2) - i(||x+iy||^2 - ||x-iy||^2)\right\} \quad \forall x, y \in V.$$

(iii) *Prove that a normed vector space* V *is an inner product space iff its norm satisfies the parallelogram identity*

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2) \quad \forall x, y \in V.$$

The following two exercise are intended to practice computation with inner products.

Exercise 2 (5 points). *Let* $\mathcal{U} := \{f \in L^2(0,1) : f(t) = at + b, a, b \in \mathbb{C}\}$ *and let* $g(t) := t^3$. *Find the projection of* g *on the subspace* $\mathcal{U}$.

Exercise 3 (5 points). *Let* $\{e_j\}_{j=1}^n$ *be an orthonormal set on a Hilbert space* $\mathcal{H}$ *and let* $x \in \mathcal{H}$. *Define* $f : \mathbb{C}^n \to \mathbb{R}$ *by*

$$f(c) := \left\| x - \sum_{j=1}^{n} c_j e_j \right\|^2, \quad c := (c_1, \ldots, c_n).$$

For which $c \in \mathbb{C}^n$ *does this function achieve its minimum?*

In the lecture we defined the orthogonal complement for a subspace of an inner product space $\mathcal{H}$. This definition extends naturally to any subset M of the space, i.e.

$$M^\perp := \{x \in \mathcal{H} : (x,\, y) = 0 \quad \text{for all} \quad y \in M\}.$$

In the following exercise you are asked to prove several important properties of orthogonal complements.

Exercise 4 (5 points)**.** *Let $\mathcal{H}$ be an inner product space and let $L, M \subset \mathcal{H}$ be non-empty. Prove the following statements:*

(i) $M^\perp$ *is a closed subspace of* $\mathcal{H}$.

(ii) $L \subset M$ *implies* $L^\perp \supset M^\perp$.

(iii) $M \cap M^\perp \subset \{0\}$, $M \subset (M^\perp)^\perp$ *and* $M^\perp = ((M^\perp)^\perp)^\perp$.

(iv) $M^\perp = \left(\overline{\mathrm{span}\, M}\right)^\perp$, *where* $\mathrm{span}\, M$ *denotes the set of all finite linear combinations of elements of* M.

And we end up this exercise sheet with two more questions involving subsets of inner product spaces.

Exercise 5 (5 points)**.** *Let* $\mathcal{H} = C([-1,1])$ *be equipped with* $(f,g) := \int_{-1}^{1} \overline{f(x)} g(x)\, dx$. *Compute the orthogonal complement of the set* $M := \{f \in \mathcal{H} \,|\, f(x) = f(-x)\, \forall x \in [0,1]\}$.

Exercise 6 (5 points)**.** *Let* $M := \{x \in c_c : \sum_{n=1}^{\infty} x_n = 0\}$, *where* c_c *is the space of finitely supported sequences, i.e.* $c_c := \{x \in \ell^\infty : x_n \neq 0$ *for at most finitely many* $n \in \mathbb{N}\}$. *Prove that* M *is dense in* ℓ^2.

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