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Functional Analysis II

Assignment 14

Problem 53. Let A be a symmetric operator on a Hilbert space $\mathcal{H}$ such that its domain $\mathcal{D}(A)$ contains an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. Prove:

(*i*) A is essentially self-adjoint.

(*ii*) $\sigma(\overline{A})$ is the closure of the set of the eigenvalues of A.

Problem 54 (Dirichlet and Neumann Laplacian in one dimension). On the Hilbert space $L^2([0,1])$ consider the densely defined operators A_D and A_N given by

$$\mathcal{D}(A_D) = \{\psi \in C^2([0,1]) \,|\, \psi(0) = 0 = \psi(1)\}, \quad \mathcal{D}(A_N) = \{\psi \in C^2([0,1]) \,|\, \psi'(0) = 0 = \psi'(1)\}$$
$$A_D\psi = -\psi'', \qquad A_N\psi = -\psi''.$$

(*i*) Prove that A_D and A_N are symmetric.

(*ii*) Prove that A_D is essentially self-adjoint and find $\sigma(\overline{A_D})$.

(*iii*) Prove that A_N is essentially self-adjoint and find $\sigma(\overline{A_N})$.

(*iv*) Let $A_{D,N}$ be given by $\mathcal{D}(A_{D,N}) = \mathcal{D}(A_D) \cap \mathcal{D}(A_N)$ and $A_{D,N}\psi = -\psi''$. Prove that $A_{D,N}$ has at least two distinct self-adjoint extensions.

Problem 55. Show that if A is a densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ then $U(t) := e^{itA}$ defines a strongly continuous unitary group $\{U(t)\}_{t\in\mathbb{R}}$ and $U'(t) = iAU(t)$ holds in the strong operator topology.

Problem 56 (Stone's theorem). Let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous unitary group on a Hilbert space $\mathcal{H}$.

(*i*) Let $\mathcal{D}$ be the set of $\varphi_f \in \mathcal{H}$ of the form $\varphi_f = \int_{-\infty}^{\infty} f(t)\, U(t)\varphi\, dt$ for some $\varphi \in \mathcal{H}$ and $f \in C_0^\infty(\mathbb{R})$, where the integral is a Hilbert space valued Riemann integral. Prove that $\mathcal{D}$ is dense in $\mathcal{H}$.

(*ii*) Prove that the operator A given by $\mathcal{D}(A) = \mathcal{D}$ and $A\varphi_f := -i\varphi_{(-f')}$ is symmetric.

(*iii*) Prove that both A and $U(t)$ leave $\mathcal{D}$ invariant and commute on $\mathcal{D}$.

(*iv*) Prove that A is essentially self-adjoint.

(*v*) Prove that $U(t) = e^{it\overline{A}}$.
[*Hint:* Set $w(t) := U(t)\varphi - e^{it\overline{A}}\varphi$ for $\varphi \in \mathcal{D}$ and compute $\frac{d}{dt}\|w(t)\|^2$.]

For more details please visit http://www.math.lmu.de/~gottwald/14FA2/