Prof. T. Ø. Sørensen PhD
S. Gottwald

Winter term 2014/15
Dec 04, 2014

Functional Analysis II

Assignment 9

Problem 33. Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Assume that $\mathbb{O} \leqslant A \leqslant B$ and $\lambda > 0$. Prove:

(*i*) $A+\lambda\mathbb{I}$ and $B+\lambda\mathbb{I}$ are invertible, and $(B+\lambda\mathbb{I})^{-1} \leqslant (A+\lambda\mathbb{I})^{-1}$ for all $\lambda > 0$.

(*ii*) If A is invertible then B is invertible too, and $B^{-1} \leqslant A^{-1}$.

Problem 34 (Stone's formula). Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove for $a, b \in \mathbb{R}$, $a < b$, that

$$\frac{1}{\pi i} \int_a^b \Big((A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1} \Big)\, d\lambda \;\longrightarrow\; \chi_{[a,b]}(A) + \chi_{(a,b)}(A) \qquad (*)$$

strongly, as $\varepsilon \to 0^+$. Both sides are defined via the measurable functional calculus.

Problem 35. Let $A \in \mathcal{B}(L^2([0,1]))$ be such that $A \geqslant 0$ and

$$A^{2014} e^A f(x) = e f(x) + e \int_0^x f(y)\, dy\,,$$

where $e := \exp(1)$. Find $\sigma(A)$. Moreover, calculate the limit $(*)$ in Problem 34 for $a = 0$, $b = 1$, and the operator A from this exercise, and show that it is even in operator norm.

Problem 36 (Discrete Laplacian). Let $-\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by

$$(-\Delta x)_n := \sum_{\substack{m\in\mathbb{Z},\\ |m-n|=1}} (x_n - x_m) \quad \forall n \in \mathbb{Z}.$$

(*i*) Show that $-\Delta$ is bounded and self-adjoint.

(*ii*) Show that $\mathbb{O} \leqslant -\Delta \leqslant 4\mathbb{I}$

(*iii*) Compute $\|-\Delta\|$.

(*iv*) Determine $\sigma(-\Delta)$.

(*v*) Find a measure space $(\mathcal{M}, \mu)$, an isomorphism $U : \ell^2(\mathbb{Z}) \to L^2(\mathcal{M}, \mu)$, and a function $F : \mathcal{M} \to \mathbb{R}$ such that $U(-\Delta)U^{-1}$ is the operator of multiplication by F.

For more details please visit http://www.math.lmu.de/~gottwald/14FA2/