LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis II

Assignment 8

Problem 29. Let $\mathcal{H}$ be a Hilbert space and $A = A^* \in \mathcal{B}(\mathcal{H})$. Prove:

(*i*) $A \leqslant \|A\| \, \mathbb{I}$.

(*ii*) If $A \geqslant \mathbb{O}$ then $\sigma(A) \subset [0, \|A\|]$.

(*iii*) If $\sigma(A) \subset [0, R]$ for some $R > 0$, then $\mathbb{O} \leqslant A \leqslant R \, \mathbb{I}$.

Problem 30. Let $\mathcal{H}$ be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. Prove:

(*i*) There exist unique self-adjoint operators $R_A, I_A \in \mathcal{B}(\mathcal{H})$ such that $A = R_A + iI_A$.

(*ii*) A is normal iff $[R_A, I_A] := R_A I_A - I_A R_A = \mathbb{O}$.

(*iii*) A is unitary iff A is normal and $R_A^2 + I_A^2 = \mathbb{I}$.

(*iv*) If $T = T^*$ and $\|T\| \leqslant 1$, then $U := T + i\sqrt{\mathbb{I} - T^2}$ is unitary and $T = \frac{1}{2}(U + U^*)$.

(*v*) There exist unitary operators $U_1, \dots, U_4$ and $a_1, \dots, a_4 \in \mathbb{C}$ such that

$$A = a_1 U_1 + a_2 U_2 + a_3 U_3 + a_4 U_4 \,,$$

and $|a_j| \leqslant \|A\|/2$ for all j.

Problem 31. Let $\mathcal{H}$ be a Hilbert space and let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove:

(*i*) If $A \leqslant B$ then $C^*AC \leqslant C^*BC$ for all $C \in \mathcal{B}(\mathcal{H})$.

(*ii*) If $\mathbb{O} \leqslant A \leqslant B$ then $\|A\| \leqslant \|B\|$.

(*iii*) If $A \geqslant \mathbb{O}$, then A is invertible iff $A \geqslant c \, \mathbb{I}$ for some $c > 0$.

Problem 32. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be normal. Prove:

(*i*) $N(T) = N(T^*)$.

(*ii*) $\overline{R(T)} = \overline{R(T^*)}$, and if $R(T)$ is closed then $R(T^*) = R(T)$.

(*iii*) If T has a one-sided inverse then T is invertible.

(*iv*) The right shift $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ can not be written as the product of a finite number of normal operators in $\ell^2(\mathbb{N})$.

For more details please visit http://www.math.lmu.de/~gottwald/14FA2/