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Functional Analysis II
Assignment 6

Problem 21. Let $d \geqslant 1$ and $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ let

$$Tf(x) := \int_{\mathbb{R}^d} k(x,y) f(y)\, dy \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

(i) Prove that this defines $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ and find an upper bound for $\|T\|$.

(ii) Prove that T is compact.

(iii) Prove for any orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(\mathbb{R}^d)$ that $\sum_{n=1}^{\infty} \|T\varphi_n\|_2^2 = \|k\|_2^2$.

(iv) Prove that $\dim N(T-I) \leqslant \|k\|_2^2$.

Problem 22.

(i) Find an example of a bounded operator T on a Hilbert space $\mathcal{H}$ such that

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| \; < \; \|T\| \,.$$

(ii) Let $R > 0$. Find an example of a 2×2 matrix A with $\sigma(A) = \{0\}$ and $\|A\| \geqslant R$.

Problem 23.

(i) Show that $|A+B| \leqslant |A| + |B|$ is *not* true for arbitrary compact operators A and B.

(ii) Prove for compact operators A, B on a Hilbert space $\mathcal{H}$ that $\frac{1}{2}|A+B|^2 \leqslant |A|^2 + |B|^2$.

Problem 24. Let $\mathcal{H}$ be a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$. Recall that U is called an *isometry* if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$, and U is called *unitary* if U is a surjective isometry. Moreover, U is called a *partial isometry* if $\|Ux\| = \|x\|$ for all $x \in N(U)^{\perp}$. Prove:

(i) U is unitary iff $U^*U = UU^* = I$.

(ii) U is an isometry iff $U^*U = I$.

(iii) If $U \neq 0$ is a partial isometry then $R(U)$ is closed and $\|U\| = 1$.

(iv) The adjoint of a partial isometry is again a partial isometry.

(v) U is a partial isometry iff U^*U is an orthogonal projection.

(vi) U is a partial isometry iff $U = UU^*U$.

For more details please visit http://www.math.lmu.de/~gottwald/14FA2/