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Functional Analysis II
Assignment 5

Problem 17. Let X be a Banach space and $T \in \mathcal{B}(X)$. Prove for any polynomial p on $\mathbb{C}$ of degree $n \geqslant 1$ that

$$\sigma(p(T)) = p(\sigma(T)).$$

Problem 18 (Square root of positive semidefinite operators). Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$ be positive semidefinite. If T were compact then the Spectral Theorem for compact operators would allow to construct the square root of T in a straightforward way (see lecture). Even though we do not have the Spectral Theorem for self-adjoint operators at our disposal yet, we can still construct $\sqrt{T}$ in this case from scratch, as will be done in this exercise. Prove:

(*i*) The power series $\sqrt{1-x} = \sum_{n=0}^{\infty} c_n x^n$ converges absolutely for $|x| \leqslant 1$, where

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n}\bigg|_{x=0} \sqrt{1-x}.$$

(*ii*) The series $S := \sqrt{\|T\|} \sum_{n=0}^{\infty} c_n \left(I - \frac{1}{\|T\|} T\right)^n$ converges in $\mathcal{B}(\mathcal{H})$, $S \geqslant 0$, and $S^2 = T$.

(*iii*) The operator $S \in \mathcal{B}(\mathcal{H})$ such that $S \geqslant 0$ and $S^2 = T$ is unique.

Problem 19 (Perturbation of the spectrum by compact operators).

(*i*) Let X be a Banach space and let $S, T \in \mathcal{B}(X)$ be such that $T - S$ is compact. Prove that $\sigma(T) \setminus \sigma_p(T) \subset \sigma(S)$. [*Hint:* Fredholm Alternative.]

(*ii*) Let $\mathcal{H}$ be a Hilbert space and let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator. Prove that

$$\sigma(U) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

(*iii*) The fact proved in (*i*) does not exclude that the two spectra may look considerably different. As an example, find a bounded operator A and a compact operator K on a Hilbert space $\mathcal{H}$ such that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}, \quad \sigma(A+K) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}.$$

Problem 20 (Volterra integral operator). Let $V : L^2([0,1]) \to L^2([0,1])$ be given by

$$(Vf)(x) := \int_0^x f(y)\, dy\,.$$

Prove the following:

(*i*) V is a well-defined, bounded operator in $L^2([0,1])$.

(*ii*) V is compact.

(*iii*) $\sigma_p(V) = \emptyset$.

(*iv*) $\sigma(V) = \{0\}$.

(*v*) $\sigma_r(V) = \emptyset$.

(*vi*) $V+V^*$ is an orthogonal projection with $\dim R(V+V^*) = 1$.