MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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Summer term 2014
Sheet 10
11.06.2014

Functional Analysis

E38 [6 points]. Let $p \in [1, \infty)$ and for $d \geqslant 1$ let $U \subset \mathbb{R}^d$ be open.

(*i*) Prove that $C_c(U)$ is separable with respect to $\|\cdot\|_p$. [*Hint:* First show that $C_c(U)$ is separable with respect to $\|\cdot\|_\infty$. For this you may use that $\mathbb{R}^d = \bigcup_{N\in\mathbb{N}}[-N, N]^d$.]

(*ii*) Conclude that $(L^p(U), \|\cdot\|_p)$ is separable.

E39 [5 points]. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, \infty)$. Let $S(\mu)$ denote the space of *integrable* simple functions (compare Definition A.19), i.e. $f : X \to \mathbb{K}$ belongs to $S(\mu)$, if there exist $N \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_N \in \mathbb{K}$, and $A_1, \ldots, A_N \in \mathcal{A}$ such that $\mu(A_k) < \infty$ for all $k = 1, \ldots, N$, and

$$f = \sum_{k=1}^{N} \alpha_k 1_{A_k} .$$

Prove that $S(\mu)$ is dense in $(L^p(X, \mu), \|\cdot\|_p)$. [*Hint:* Reduce the problem to the case of non-negative functions and use Lemma A.20.]

E40 [7 points]. Let $U \subset \mathbb{R}^d$ be open. Prove that

$$\overline{C_c(U)}^{\|\cdot\|_\infty} = C_0(U) ,$$

where

$$C_0(U) := \Big\{ f \in C(U) \,\Big|\, \forall \varepsilon{>}0\ \exists K \subset U \text{ compact, s.th. } |f(x)| < \varepsilon\, \forall x \in U \backslash K \Big\} .$$

E41 [6 points]. For $p \in [1, \infty)$ let $f, f_n \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$. Prove that the following statements are equivalent:

(*i*) $f_n \to f$ in $L^p(\mathbb{R})$ as $n \to \infty$.

(*ii*) $f_n \to f$ in $L^p([-N, N])$ as $n \to \infty$ for all $N \in \mathbb{N}$, and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

Please hand in your solutions until next **Wednesday (18.06.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/