Mathematisches Institut
der Universität München
Prof. Dr. Peter Müller

Summer term 2014
Sheet 8
28.05.2014

Functional Analysis

E29 [4 points]. Let $(X, \mathcal{A}, \mu)$ be a measure space and $p \in [2, \infty)$. Prove that

$$\|f+g\|_p^p + \|f-g\|_p^p \leqslant 2^{p-1}\big(\|f\|_p^p + \|g\|_p^p\big)$$

for all $f, g \in L^p(X, \mu)$. [*Hint:* Show that $a^p+b^p \leqslant (a^2+b^2)^{p/2}$ for all $a, b \geqslant 0$ and use convexity of the function $\mathbb{R}_+ \to \mathbb{R}, x \mapsto x^{p/2}$.]

E30 [10 points]. For $n \in \mathbb{Z}$ and $x \in [-1, 1]$ let $e_n(x) := \frac{1}{\sqrt{2}}\, e^{i\pi nx}$. Prove the following statements:

(*i*) $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2([-1, 1])$. [*Hint:* You may use without proof that $C([-1, 1]) \subset L^2([-1, 1])$ is dense with respect to $\|\cdot\|_2$. Moreover, consider the space $C_p = \{f \in C([-1, 1]) \,|\, f(-1) = f(1)\} \cong C(S^1)$ and use the Stone-Weierstraß theorem to show that $\operatorname{span}\{e_n\}_{n\in\mathbb{Z}}$ is dense in $(C_p, \|\cdot\|_\infty)$.]

(*ii*) By (*i*), for any $f \in L^2[-1, 1]$ there exists a set $\{c_n\}_{n\in\mathbb{Z}}$ of coefficients such that $f = \sum_{n\in\mathbb{Z}} c_n e_n$, which we call the *Fourier series of* f. For $n \in \mathbb{Z}$ give an expression for the *Fourier coefficient* c_n in terms of f, and also express $\|f\|_2$ in terms of $\{c_n\}_{n\in\mathbb{Z}}$.

(*iii*) Calculate $\sum_{n=1}^{\infty} n^{-2}$ by means of the Fourier coefficients of $f : [-1, 1] \to \mathbb{R}, x \mapsto x$.

E31 [4 points]. Let $(0, 1) \subset \mathbb{R}$ and $\mathbb{R}$ be equipped with the Lebesgue measure on their Borel σ-algebras, and let $1 \leqslant p, q \leqslant \infty$. Prove:

(*i*) If $p < q$, then $L^q((0, 1)) \subsetneq L^p((0, 1))$.

(*ii*) If $p \neq q$, then $L^q(\mathbb{R}) \not\subset L^p(\mathbb{R})$.

E32 [6 points]. Let λ^d be the Lebesgue measure on $\mathbb{R}^d$, let $p \in [1, \infty)$ and let $M \subset L^p(\mathbb{R}^d)$ be relatively compact. Prove the following statements:

(*i*) M is bounded.

(*ii*) $\displaystyle\lim_{y\to 0} \sup_{f\in M} \int_{\mathbb{R}^d} |f(x+y) - f(x)|^p \, d\lambda^d(x) = 0$

(*iii*) $\displaystyle\lim_{r\to\infty} \sup_{f\in M} \int_{|x|>r} |f(x)|^p \, d\lambda^d(x) = 0$

[*Hint:* For (*ii*) and (*iii*) cover M by balls and approximate the centers by simple functions. *Remark:* One can also show that any $M \subset L^p(\mathbb{R}^d)$ satisfying (*i*), (*ii*) and (*iii*) is relatively compact.]

Please hand in your solutions until next **Wednesday (04.06.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/