Mathematisches Institut
der Universität München
Prof. Dr. Peter Müller

Summer term 2014
Sheet 6
14.05.2014

Functional Analysis

E21 [8 points]. Let $p \in (1, \infty)$ and let q be the *Hölder conjugate* of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in \ell^p$ and $y \in \ell^q$ let $\langle y, x \rangle := \sum_{n \in \mathbb{N}} y_n x_n$ (defined due to Hölder's inequality, compare Lemma 1.27). Prove the following statements:

(*i*) $\|x\|_p = \sup\limits_{0 \neq y \in \ell^q} \dfrac{|\langle y, x \rangle|}{\|y\|_q}$ for all $x \in \ell^p$.

(*ii*) $\|T\| = \sup\limits_{\substack{0 \neq y \in \ell^q \\ 0 \neq x \in \ell^p}} \dfrac{|\langle y, Tx \rangle|}{\|y\|_q \|x\|_p}$ for every bounded linear operator $T : \ell^p \to \ell^p$.

E22 [6 points]. Let p, q be defined as in E21. For $j, k \in \mathbb{N}$ let $c_{j,k} \in \mathbb{C}$ be such that

$$a := \sup_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}} |c_{j,k}| < \infty \quad \text{and} \quad b := \sup_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} |c_{j,k}| < \infty .$$

Prove that $(Tx)_j := \sum_{k \in \mathbb{N}} c_{j,k} x_k$ defines a linear operator $T : \ell^p \to \ell^p$ with $\|T\| \leqslant a^{1/p} b^{1/q}$.

E23 [4 points]. Let c_0 (compare T5) be equipped with the norm $\| \cdot \|_\infty$. Find a bounded linear operator $T : c_0 \to c_0$ such that

$$\|Tx\|_\infty < \|T\|$$

for all $x \in \partial B_1(0) = \{x \in c_0 : \|x\|_\infty = 1\}$.

E24 [6 points]. As in E23, let c_0 be equipped with $\| \cdot \|_\infty$. Prove the following statements:

(i) The family $\{e_n\}_{n \in \mathbb{N}}$, where $(e_n)_k := \delta_{nk}$ for $k \in \mathbb{N}$, forms a Schauder basis of c_0.

(ii) $c_0^* \cong \ell^1$ (i.e. c_0^* and ℓ^1 are isometrically isomorphic)

(iii) c_0^* can be identified with a subspace of $(\ell^\infty)^*$, in the sense that there exists a linear isometry $J : c_0^* \to (\ell^\infty)^*$.

Please hand in your solutions until next **Wednesday (21.05.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/