Mathematisches Institut
der Universität München
Prof. Dr. Peter Müller

Summer term 2014
Sheet 3
23.04.2014

Functional Analysis

E9 [4 points]. Let X be a compact topological space and let Y be a Hausdorff space.

(i) Show that any continuous bijection $f : X \to Y$ is a homeomorphism.

(ii) Find a counterexample for (i) when Y is not a Hausdorff space.

E10 [6 points]. Let $f : X \to Y$ be a continuous map between topological spaces X, Y. Prove the following:

(i) If X is compact and $Y = \mathbb{R}$ (equipped with the standard topology) then f attains its minimum and its maximum.

(ii) If X and Y are metric spaces and X is compact then f is uniformly continuous.

E11 [7 points]. Let $I \neq \emptyset$ be an index set and for each $\alpha \in I$ let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. Let

$$X := \bigtimes_{\alpha \in I} X_\alpha := \Big\{ x : I \to \bigcup_{\alpha \in I} X_\alpha \,\Big|\, x(\alpha) \in X_\alpha \Big\}.$$

be equipped with the product topology $\mathcal{T}$ (see Definition 1.9). Prove the following:

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ converges in $(X, \mathcal{T})$ to $x \in X$ if and only if for each $\alpha \in I$ the sequence $(x_n(\alpha))_{n\in\mathbb{N}}$ converges to $x(\alpha)$ in $(X_\alpha, \mathcal{T}_\alpha)$. Thus, the product topology coincides with the topology of pointwise convergence.

(ii) Consider $I = \mathbb{N}$ and $X_\alpha = \mathbb{C}$ for all $\alpha \in I$. Find a sequence[1] $(a^{(n)})_{n\in\mathbb{N}}$ in $\ell^1 \subset X$ such that $a^{(n)} \to 0$ with respect to $\mathcal{T}$ but $\|a^{(n)}\|_1 \to \infty$ as $n \to \infty$.

(iii) Show that $\bar{B}_1^{\ell^\infty}(0) = \bigtimes_{\mathbb{N}} \bar{B}_1^{\mathbb{C}}(0)$ is compact in the product topology, where $\bar{B}_r^X(z)$ denotes the closed ball of radius $r > 0$ around $z \in X$ in the metric space (X, d) (compare E2(i)). Argue that $\bar{B}_1^{\ell^\infty}(0)$ is not compact in the metric topology of ℓ^∞ without considering an explicit sequence (in contrast to Warning 1.36 of the lecture).

E12 [7 points]. A metric space X is called *totally bounded* if for all $\varepsilon > 0$ there exists a finite set F and sets U_j with $\operatorname{diam} U_j < \varepsilon$ for all $j \in F$, such that $X = \bigcup_{j\in F} U_j$. Prove that for a metric space X the following properties are equivalent[2]:

(1) X is compact.

(2) Every sequence $(x_n)_{n\in\mathbb{N}}$ in X has at least one accumulation point.

(3) X is totally bounded and complete.

Give an example of a metric space that is bounded but not totally bounded.

Please hand in your solutions until next **Wednesday (30.04.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/

[1] Note that $a^{(n)} \in \ell^1$ for each n, i.e. $(a^{(n)})_{n\in\mathbb{N}}$ is a sequence of sequences.

[2] The equivalence of (1) and (2) is known from the lecture.