E24 [6 points]. As in E23, let c_0 be equipped with $\|\cdot\|_\infty$. Prove the following statements:

(*i*) The family $\{e_n\}_{n\in\mathbb{N}}$, where $(e_n)_k := \delta_{nk}$ for $k \in \mathbb{N}$, forms a Schauder basis of c_0.

Proof. For any $x = (x_k)_{k\in\mathbb{N}} \in c_0$ we have

$$\lim_{n\to\infty}\left\| x - \sum_{k=1}^{n} x_k e_k \right\|_\infty = \lim_{n\to\infty} \sup_{k>n} |x_k| = \limsup_{n\to\infty} |x_n| = 0\,.$$

Hence $\{e_n\}_n$ is a Schauder basis of $(c_0, \|\cdot\|_\infty)$. □

(*ii*) $c_0^* \cong \ell^1$ (i.e. c_0^* and ℓ^1 are isometrically isomorphic)

Proof. Since for each $y \in \ell^1$ and $x \in c_0$, we have $\sum_n |y_n||x_n| \leqslant \|y\|_1\|x\|_\infty$ we may define the linear map

$$I : \ell^1 \to c_0^*\,,\ y \mapsto f_y\,, \quad f_y(x) := \sum_n y_n x_n\,,$$

where $\|f_y\|_* \leqslant \|y\|_1$. Next, for $y \in \ell_1$ and fixed $N \in \mathbb{N}$ let

$$\tilde{x}_n := \tfrac{|y_n|}{y_n} \ \text{ if } \ n \leqslant N \wedge y_n \neq 0\,, \quad \tilde{x}_n := 0 \ \text{ otherwise.}$$

It follows that $\tilde{x} \in c_0$ and

$$\|f_y\|_* = \|f_y\|_*\|\tilde{x}\|_\infty \ \geqslant\ |f_y(\tilde{x}_n)| = \sum_n y_n\tilde{x}_n = \sum_{n=1}^{N} |y_n|\,,$$

in particular $\|y\|_1 \leqslant \|f_y\|_*$. Hence $\|Iy\|_* = \|y\|_1$, i.e. I is an isometry. It remains to show that I is surjective. For this, let $f \in c_0^*$ and define $y_n := f(e_n)$, where $\{e_n\}_n$ is the Schauder basis of c_0 defined in (*i*). It follows for any $x \in c_0$ that

$$\sum_{n=1}^{\infty} f(e_n)x_n \overset{f \text{ linear}}{=} \lim_{N\to\infty} f\left(\sum_{n=1}^{N} x_n e_n\right) \overset{f \text{ cont.}}{=} f\left(\sum_{n=1}^{\infty} x_n e_n\right) = f(x)\,. \qquad (*)$$

Moreover, by the same argument as above, we have $\|y\|_1 \leqslant \|f\|$, because $(*)$ shows that $f_y = f$ if $y_n = f(e_n)$. Thus $y \in \ell^1$ and from $(*)$ it follows that $Iy = f$. Hence I is surjective, and therefore (since any isometry is injective) it is an isometric isomorphism between ℓ^1 and c_0^*. □

(*iii*) c_0^* can be identified with a subspace of $(\ell^\infty)^*$, in the sense that there exists a linear isometry $J : c_0^* \to (\ell^\infty)^*$.

Proof. For $f \in c_0^*$, it follows from the proof of (*i*) that $y_n := f(e_n)$ defines $y \in \ell^1$ with $\|y\|_1 = \|f\|_*$. Hence we may define $J : c_0^* \to (\ell^\infty)^*$ by $Jf(x) := \sum_n f(e_n)x_n$ $\forall x \in \ell^\infty$, since $\sum_n |f(e_n)||x_n| \leqslant \|y\|_1\|x\|_\infty$, i.e. $\|Jf\|_* \leqslant \|y\|_1 = \|f\|_*$. Moreover, note that by $(*)$ we have $Jf(x) = f(x)$ for all $x \in c_0 \subset \ell^\infty$ and therefore

$$\|Jf\|_* = \sup_{0\neq x\in\ell^\infty} \frac{|Jf(x)|}{\|x\|_\infty} \geqslant \sup_{0\neq x\in c_0} \frac{|Jf(x)|}{\|x\|_\infty} = \sup_{0\neq x\in c_0} \frac{|f(x)|}{\|x\|_\infty} = \|f\|_*$$

Hence $\|Jf\|_* = \|f\|_*$ for all $f \in (\ell^\infty)^*$ and the claim follows. □