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Solution 5
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Functional Analysis

E17 [7 points]. Let X be a normed space with $\dim X = \infty$ and let $B \subset X$ be a Hamel basis of X.

(*i*) Let $L \subset X$ be a finite-dimensional subspace. Prove that L is nowhere dense in X.

Proof. Since L is closed (see e.g. E15), we need to show that $\mathring{L} = \emptyset$. For this, let $x \in L$ and $\varepsilon > 0$. We conclude that $B_\varepsilon(x) \not\subset L$, which proves that L has no interior points. There is $b \in B$ such that $b \notin L$ (otherwise $\dim L = \infty$) and therefore $y := x + \frac{\varepsilon}{2}\frac{b}{\|b\|}$ does not belong to L as well (otherwise $b = \frac{2\|b\|}{\varepsilon}(y - x) \in L$). But $y \in B_\varepsilon(x)$, since $\|x - y\| = \frac{\varepsilon}{2} < \varepsilon$. □

(*ii*) Show that if X is complete, then B is uncountable.

Proof. Assume that B was countable, i.e. $B = \{b_i\}_{i\in\mathbb{N}}$. For $N \in \mathbb{N}$ let A_N be the linear subspace of X given by $A_N = \operatorname{span}\{b_1, \dots, b_N\} = \{\sum_{n=1}^N \alpha_n b_n \mid \alpha_i \in \mathbb{K}\}$. It follows from (*i*), that A_N is nowhere dense. Moreover, since B is a Hamel basis, for each $x \in X$ there exists $N_x \in \mathbb{N}$ such that $x \in A_{N_x}$, in particular $X = \bigcup_{N\in\mathbb{N}} A_N$. Hence X is meagre, which contradicts (a corollary to) Baire's theorem, which says that X has to be non-meagre if it is complete (and non-empty). □

(*iii*) Let $\mathcal{P}$ be the linear space of all real-valued polynomials on $\mathbb{R}$ and for $p \in \mathcal{P}$, given by $p(t) = \sum_{k=0}^n a_k t^k$, define $\|p\| := \sum_{k=0}^n |a_k|$. Prove that $(\mathcal{P}, \|\cdot\|)$ is a normed space and argue whether or not it is complete.

Proof. The fact that $\|\cdot\|$ is a norm on $\mathcal{P}$ follows directly from the properties of the absolute value on $\mathbb{R}$ and the linearity of the sum. If M denotes the set of all monomials $m_k : \mathbb{R} \to \mathbb{R}, t \mapsto t^k$, then M is a Hamel basis of $\mathcal{P}$, since any $p \in \mathcal{P}$ can be written as $p = \sum_{k=0}^n a_k m_k$ for some $n \in \mathbb{N}$. Since $M = \{m_k\}_{k=0}^\infty$ is countable, it follows from (*ii*) that $\mathcal{P}$ cannot be complete. □