Essential dimension of Hermitian spaces

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Abstract

Given an hermitian space we compute its essential dimension, Chow motive and prove its incompressibility in certain dimensions.

The notion of an *essential dimension* $\dim_{es}$ is an important birational invariant of an algebraic variety X which was introduced and studied by N. Karpenko, A. Merkurjev, Z. Reichstein, J.-P. Serre and others. Roughly speaking, it is defined to be the minimal possible dimension of a rational retract of X. In particular, if it coincides with the usual dimension, then X is called *incompressible*.

In general, this invariant is very hard to compute. As a consequence, there are only very few known examples of incompressible varieties: certain Severi-Brauer varieties and projective quadrics. In the present paper we provide new examples of incompressible varieties: *Hermitian quadrics* of dimensions $2^r - 1$. We also give an explicit formula for the essential dimension of a Hermitian form in the sense of O. Izhboldin, hence, providing a Hermitian version of the result of Karpenko-Merkurjev [KM03]. At the end we discuss the relations with Higher forms of Rost motives of Vishik [Vi00].

We follow the notation of [Kr07]. Let F be a base field of characteristic not 2 and let L/F be a quadratic field extension. Let (W, h) be a non-degenerate L/F-Hermitian space of rank n and let q be the quadratic form associated to the hermitian form h via $q(v) = h(v, v)$, $v \in W$. We call q the underlying quadratic form of h.

The main objects of our study are the following two smooth projective varieties over F:

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- the variety $V(q)$ of q-isotropic F-lines in W, i.e. a projective *quadric*;
- the variety $V(h)$ of h-isotropic L-lines in W called a *Hermitian quadric*.

Observe that $V(q)$ has dimension $(2n-2)$ and $V(h)$ is a $(2n-3)$-dimensional projective homogeneous variety under the action of the unitary group associated with h. It is also a twisted form of the *incidence variety* that is the variety of flags consisting of a dimension one and codimension one linear subspaces in an n-dimensional vector space (see [MPW98, pp. 172-173]).

The forms q and h are closely related by the following celebrated result of Milnor-Husemoller (see [Le79]):

> A quadratic form q on an F-vector space W is the underlying form of a Hermitian form over a quadratic field extension $L = F(\sqrt{a})$ if and only if $\dim W = 2n$, q_L is hyperbolic, and $\det q = (-a)^n \mod F^2$.

In this case $q \simeq \langle 1, -a\rangle \otimes q'$ for some form q'.

1. Incompressibility A smooth projective F-variety X is called *incompressible* if any rational map $X \dashrightarrow X$ is dominant. The basic examples of such varieties are anisotropic quadrics of dimensions $2^r - 1$ and Severi-Brauer varieties of division algebras of prime degrees.

Theorem (A)**.** *Assume that the variety $V(h)$ is anisotropic, i.e., has no F-rational points, and $\dim V(h) = 2^r - 1$ for some $r > 0$. Then $V(h)$ is incompressible.*

Proof. The key idea is that a Hermitian quadric which is purely a geometric object can be viewed as a twisted form of a *Milnor hypersurface* M — a topological object, namely, a generator of the Lazard ring of *algebraic cobordism* of M. Levine and F. Morel [LM].

More precisely, by [LM, 2.5.3] the variety M is the zero divisor of the line bundle $\mathcal{O}(1) \otimes \mathcal{O}(1)$ on $\mathbb{P}^{n-1}_F \times \mathbb{P}^{n-1}_F$, i.e. it is given by the equation

$$\sum_{i=0}^{n-1} x_i y_i = 0, \qquad (1)$$

where $[x_0 : \dots x_{n-1}]$ and $[y_0 : \dots : y_{n-1}]$ are the projective coordinates of the first and the second factor respectively.

On the other hand, the Hermitian quadric $V(h)$ is a twisted form of the incidence variety $X = \{W_1 \subset W_{n-1}\}$, where $\dim W_i = i$. Taking $[x_0 : \ldots : x_{n-1}] = W_1$ and $[y_0 : \ldots : y_{n-1}]$ to be a normal vector to W_{n-1} we obtain that X is given by the same equation (1), therefore, $X \simeq M$.

By [Me02, Prop.7.2] we obtain the following explicit formula for the Rost characteristic number η_2 of M

$$\eta_2(M) := \frac{c_{\dim M}(-T_M)}{2} = \frac{1}{2}\binom{2(n-1)}{n-1} \mod 2.$$

It has the following property:

$$\eta_2(M) \equiv 1 \mod 2 \iff \dim M = 2^r - 1 \text{ for some } r > 0. \tag{2}$$

Since η_2 is invariant under field extensions, $\eta_2(M) = \eta_2(V(h))$.

We apply now the standard arguments involving the Rost degree formula (see [Me03, §7]). Let $f\colon V(h) \dashrightarrow V(h)$ be a rational map. By the degree formula:

$$\eta_2(V(h)) \equiv \deg f \cdot \eta_2(V(h)) \mod n_{V(h)}, \tag{3}$$

where $n_{V(h)}$ is the greatest common divisor of degrees of all closed points of $V(h)$. Since $V(h)$ becomes isotropic over L, $n_{V(h)} = 2$.

Assume now that $\dim(V(h)) = 2^r - 1$ for some $r > 0$. Then, by (2) $\eta_2(V(h)) \equiv 1$ and by (3) $\deg f \neq 0$ which means that f is dominant. This finishes the proof of the theorem. □

2. Essential dimension Following O. Izhboldin we define the *essential dimension* of a Hermitian space (W, h) as

$$\dim_{es}(h) := \dim V(h) - i(q) + 2,$$

where $i(q)$ stands for the first Witt index of the form q, i.e., for the Witt index of q over its field of rational functions (see also [KM03]). The following theorem provides a *Hermitian version* of the main result of [KM03]

Theorem (B). *Let Y be a complete F-variety with all closed points of even degree. Suppose that Y has a closed point of odd degree over $F(V(h))$. Then* $\dim_{es}(h) \leq \dim Y$. *Moreover, if* $\dim_{es}(h) = \dim Y$, *then $V(h)$ is isotropic over $F(Y)$.*

Proof. In [Kr07] D. Krashen constructed a $\mathbb{P}^1$-bundle

$$Bl_S(V(q)) \to V(h), \tag{4}$$

where $Bl_S(V(q))$ is the blow-up of the quadric $V(q)$ along the linear subspace $S = \mathbb{P}_L^{n-1}$. In particular, the function field of $V(q)$ is a purely transcendental extension of the function field of $V(h)$ of degree 1, and, therefore, our theorem follows from [KM03, Theorem 3.1]. □

Using Theorem (B) we can give an alternative proof of Theorem (A):

Another proof of (A). Let Y be the closure of the image of a rational map $V(h) \dashrightarrow V(h)$. Then by Theorem (B) the incompressibility of $V(h)$ follows from the equality $\dim_{es}(h) = \dim V(h)$. The latter can be deduced from Hoffmann's conjecture (proven in [Ka03]) if $V(h)$ is anisotropic and $\dim V(h) = 2^r - 1$. Indeed, if $\dim V(h) = 2^r - 1$, then $\dim q = 2^r + 2$. Therefore, $i(q) = 1$ or 2. But by the result of Milnor-Husemoller q is divisible by the binary form $\langle 1, -a \rangle$, hence $i(q)$ must be even. Therefore $\dim_{es}(h) = \dim V(h)$. □

3. Chow motives We follow the notation of [CM06, §6]. As a direct consequence of the fibration (4) and the Krull-Schmidt theorem proven in [CM06] we obtain the following expressions for the Chow motives of $V(q)$ and $V(h)$:

Theorem (C)**.** *There exists a motive N_h such that*

$$M(V(q)) \simeq \begin{cases} N_h \oplus N_h\{1\}, & \textit{if } n \textit{ is even;} \\ N_h \oplus M(\operatorname{Spec} L)\{n-1\} \oplus N_h\{1\}, & \textit{if } n \textit{ is odd;} \end{cases} \tag{5}$$

and

$$M(V(h)) \simeq \begin{cases} N_h \oplus \bigoplus_{i=0}^{(n-4)/2} M(\mathbb{P}_L^{n-1})\{2i+1\}, & \textit{if } n \textit{ is even;} \\ N_h \oplus \bigoplus_{i=0}^{(n-3)/2} M(\mathbb{P}_L^{n-2})\{2i+1\}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{6}$$

Observe that by the projective bundle theorem $M(\mathbb{P}_L^m) \simeq \bigoplus_{i=0}^m M(\operatorname{Spec} L)\{i\}$.

Proof. Using the $\mathbb{P}^1$-fibration (4) D. Krashen provided the following formula relating the Chow motives of $V(q)$ and $V(h)$:

$$M(V(q)) \oplus \bigoplus_{i=1}^{n-2} M(\mathbb{P}_L^{n-1})\{i\} \simeq M(V(h)) \oplus M(V(h))\{1\}. \tag{7}$$

Observe that the motives of all varieties participating in the formula (7) split over L into direct sums of twisted Tate motives $\mathbb{Z}_L$. For each such decomposition $M_L \simeq \bigoplus_{i\geq 0} \mathbb{Z}_L\{i\}^{\oplus a_i}$ we define the respective Poincaré polynomial by $P_{M_L}(t) := \sum_{i\geq 0} a_i t^i$. Using the standard combinatorial description of the cellular structure on $V(q)_L$, $V(h)_L$ and $\mathbb{P}^{n-1}_L$ (see [CM06]) we obtain the following explicit formulae:

$$P_{V(q)_L}(t) = \tfrac{(1-t^n)(1+t^{n-1})}{1-t},\ P_{V(h)_L}(t) = \tfrac{(1-t^n)(1-t^{n-1})}{(1-t)^2} \text{ and } P_{\operatorname{Spec} L}(t) = 2. \quad (8)$$

Consider the subcategory of the category of Chow motives with $\mathbb{Z}/2$-coefficients generated by $M(V(h); \mathbb{Z}/2)$. Since $V(h)$ is a projective homogeneous variety, the Krull-Schmidt theorem and the cancellation theorem hold in this subcategory by [CM06, Cor. 35]. In particular, the decompositions coming from the two sides of equation (7) have to consist of the same indecomposable summands.

Analyzing their Poincaré polynomials over L using (8) we obtain the formulae (5) and (6) for motives with $\mathbb{Z}/2$-coefficients. Finally, applying [PSZ, Thm. 2.16] for $m = 2$ we obtain the desired formulae integrally. □

4. Higher forms of Rost motives In [Vi00, Thm.5.1] A. Vishik proved that given a quadratic form q over F divisible by an m-fold Pfister form φ, that is $q = q' \otimes \varphi$ for some form q', there exists a direct summand N of the motive $M(Q_q)$ of the projective quadric Q_q associated with q such that

$$M(Q_q) \simeq \begin{cases} N \otimes M(\mathbb{P}_F^{2^m-1}), & \text{if } \dim q' \text{ is even;} \\ (N \otimes M(\mathbb{P}_F^{2^m-1})) \oplus M(Q_\varphi)\{\frac{\dim q}{2} - 2^{m-1}\}, & \text{if } \dim q' \text{ is odd.} \end{cases}$$

In view of the Milnor-Husemoller theorem mentioned in the beginning, formula (5) implies a shortened proof of Vishik's result for $m = 1$.

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