10. Functional Equation of the Zeta Function

10.1. Theorem (Functional equation of the theta function).

The theta series is defined for real $x > 0$ by

$$
\theta(x) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}.
$$

It satisfies the following functional equation

$$
\theta \left( \frac{1}{x} \right) = \sqrt{x} \theta(x) \quad \text{for all } x > 0,
$$

i.e.

$$
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / x}.
$$

Remarks. a) The theta series, as well as its derivatives, converge uniformly on every interval $[\varepsilon, \infty[$, $\varepsilon > 0$; hence $\theta$ is a $\mathcal{C}^\infty$-function on $]0, \infty[.$

b) In the theory of elliptic functions one defines more general theta functions of two complex variables. For $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$ and $z \in \mathbb{C}$ one sets

$$
\vartheta(\tau, z) := \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2\pi i n z}.
$$

For fixed $\tau$ this is an entire holomorphic function in $z$, which can be used to construct doubly periodic functions with respect to the lattice $\mathbb{Z} + \mathbb{Z} \tau$. As a function of $\tau$, it is holomorphic in the upper halfplane. The relation to the theta series of theorem 10.1 is

$$
\theta(t) = \vartheta(it, 0).
$$

Proof. For fixed $x > 0$, we consider the function $F : \mathbb{R} \to \mathbb{R},$

$$
F(t) := \sum_{n \in \mathbb{Z}} e^{-\pi (n-t)^2 x}.
$$

The series converges uniformly on $\mathbb{R}$ together with all its derivatives, hence represents a $\mathcal{C}^\infty$-function on $\mathbb{R}$. It is periodic with period 1, i.e. $F(t+1) = F(t)$ for all $t \in \mathbb{R}$. Therefore we can expand $F$ as a uniformly convergent Fourier series

$$
F(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n t}
$$

where the coefficients $c_n$ are the integrals

$$
c_n = \int_0^1 F(t) e^{-2\pi i n t} \, dt = \sum_{k \in \mathbb{Z}} \int_0^1 e^{-\pi (k-t)^2 x} e^{-2\pi i n t} \, dt.
$$
10. Functional equation

Now \( \int_0^1 e^{-\pi(k-t)^2} e^{-2\pi int} dt = \int_k^{k+1} e^{-\pi t^2} e^{-2\pi int} dt \) (substitution \( \tilde{t} = t - k \)), hence

\[
e^{-\pi t^2} e^{-2\pi int} dt.
\]

For \( n = 0 \) this is the well known integral of the Gauss bell curve

\[
c_0 = \int_{-\infty}^{\infty} e^{-\pi t^2} dt = 2 \int_0^{\infty} e^{-\pi t^2} dt = \frac{2}{\sqrt{\pi x}} \int_0^{\infty} e^{-t^2} dt
\]

\[
= \frac{1}{\sqrt{\pi x}} \int_0^{\infty} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi x}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{x}}.
\]

For general \( n \) we write

\[
-\pi t^2 x - 2\pi int = -\pi \left(t\sqrt{x} + \frac{i n}{\sqrt{x}}\right)^2 = \frac{\pi n^2}{x}.
\]

This leads to

\[
c_n = e^{-\pi n^2/x} \int_{-\infty}^{\infty} e^{-\pi (t\sqrt{x}+in/\sqrt{x})^2} dt.
\]

We will prove

\[
\int_{-\infty}^{\infty} e^{-\pi (t\sqrt{x}+in/\sqrt{x})^2} dt = \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi t^2} dt = \frac{1}{\sqrt{x}} \tag{*}
\]

Assuming this for a moment, we get

\[
F(t) = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/x} e^{2\pi int}.
\]

Setting \( t = 0 \), it follows

\[
F(0) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2/x} = \frac{1}{\sqrt{x}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/x},
\]

which is the assertion of the theorem.

It remains to prove the formula \((*)\). Using the substitution \( \tilde{t} = t\sqrt{x} \) we see that

\[
\int_{-\infty}^{\infty} e^{-\pi (t\sqrt{x}+in/\sqrt{x})^2} dt = \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi t^2} dt = \frac{1}{\sqrt{x}}
\]

With the abbreviation \( a := n/\sqrt{x} \) we have to show that

\[
\int_{-\infty}^{\infty} e^{-\pi (t+ia)^2} dt = \int_{-\infty}^{\infty} e^{-\pi t^2} dt. \tag{**}
\]

To this end we integrate the holomorphic function \( f(z) := e^{-\pi z^2} \) over the boundary of the rectangle with corners \(-R, R, R + ia, -R + ia\), where \( R \) is a positive real number.
By the residue theorem the whole integral is zero, hence
\[
\int_{-R}^{R} f(z) \, dz = \int_{-R+ia}^{R+ia} f(z) \, dz - \int_{R}^{R+ia} f(z) \, dz + \int_{-R}^{-R+ia} f(z) \, dz
\]

Now
\[
\int_{-R}^{R} f(z) \, dz = \int_{-R}^{R} e^{-\pi t^2} \, dt,
\]
\[
\int_{-R+ia}^{R+ia} f(z) \, dz = \int_{-R}^{R} e^{-\pi(t+ia)^2} \, dt,
\]
\[
\int_{\pm R+ia}^{\pm R} f(z) \, dz = i \int_{0}^{a} e^{-\pi(R^2-t^2)+2\pi It} \, dt = i e^{-\pi R^2} \int_{0}^{a} e^{\pi t^2+2\pi It} \, dt.
\]

We have the estimate
\[
\left| \int_{\pm R}^{\pm R+ia} f(z) \, dz \right| \leq e^{-\pi R^2 |a|} e^{\pi |a|^2},
\]
which tends to 0 as \( R \to \infty \). This implies
\[
\lim_{R \to \infty} \int_{-R}^{R} e^{-\pi t^2} \, dt = \lim_{R \to \infty} \int_{-R}^{R} e^{-\pi(t+ia)^2} \, dt,
\]
which proves (**) and therefore (*). This completes the proof of the functional equation of the theta function.

10.2. Corollary. The theta function \( \theta(x) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2x} \) defined in the preceding theorem satisfies
\[
\theta(x) = O\left( \frac{1}{\sqrt{x}} \right) \quad \text{as} \quad x \downarrow 0.
\]

10.3. Proposition. For all \( s \in \mathbb{C} \) with \( \Re(s) > 1 \) one has
\[
\Gamma\left( \frac{s}{2} \right) \zeta(s) = \pi^{s/2} \int_{0}^{\infty} t^{s/2} \left( \sum_{n=1}^{\infty} e^{-\pi n^2 t} \right) \frac{dt}{t}.
\]
Remark. The function

\[ \psi(t) := \sum_{n=1}^{\infty} e^{-\pi n^2 t} \]

decreases exponentially as \( t \to \infty \). One has \( \theta(t) = 1 + 2\psi(t) \), hence \( \psi(t) = \frac{1}{2}(\theta(t) - 1) \), so corollary 10.2 implies

\[ \psi(t) = O\left(\frac{1}{\sqrt{t}}\right) \quad \text{for} \quad t \searrow 0. \]

This shows that the integral exists for \( \text{Re}(s) > 1 \).

Proof. We start with the Euler integral for \( \Gamma(s/2) \),

\[ \Gamma\left(\frac{s}{2}\right) = \int_{0}^{\infty} t^{s/2} e^{-t} \frac{dt}{t}, \]

and apply the substitution \( \tilde{t} = \pi n^2 t \), where \( n \in \mathbb{N}_1 \). Since \( d\tilde{t}/\tilde{t} = dt/t \), we get

\[ \Gamma\left(\frac{s}{2}\right) = n^s \pi^{s/2} \int_{0}^{\infty} t^{s/2} e^{-\pi n^2 t} \frac{dt}{t}. \]

For \( \text{Re}(s) > 1 \) we have

\[ \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^{\infty} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \sum_{n=1}^{\infty} \pi^{s/2} \int_{0}^{\infty} t^{s/2} e^{-\pi n^2 t} \frac{dt}{t} = \pi^{s/2} \int_{0}^{\infty} t^{s/2} \left( \sum_{n=1}^{\infty} e^{-\pi n^2 t} \right) \frac{dt}{t}. \]

The interchange of summation and integration is allowed by the theorem of majorized convergence for Lebesgue integrals.

10.4. Theorem (Functional equation of the zeta function).

a) The function

\[ \xi(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s), \]

which is a meromorphic function in \( \mathbb{C} \), satisfies the functional equation

\[ \xi(1 - s) = \xi(s). \]

b) For the zeta function itself one has

\[ \zeta(1 - s) = 2^{1-s} \pi^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \]
Proof. By the preceding theorem

\[ \xi(s) = \int_0^\infty t^{s/2} \psi(t) \frac{dt}{t} \quad \text{with} \quad \psi(t) = \sum_{n=1}^\infty e^{-\pi n^2 t}. \]

The functional equation of the theta function implies for \( \psi(t) = \frac{1}{2}(\theta(t) - 1) \)

\[ \psi(t) = t^{-1/2} \psi(1/t) - \frac{1}{2}(1 - t^{-1/2}). \]

We substitute this expression in the integral from 0 to 1:

\[ \int_0^1 t^{s/2} \psi(t) \frac{dt}{t} = \int_0^1 t^{(s-1)/2} \psi\left(\frac{1}{t}\right) \frac{dt}{t} + \frac{1}{2} \int_0^1 (t^{(s-1)/2} - t^{s/2}) \frac{dt}{t}. \]

The last integral can be evaluated explicitly (recall that \( \text{Re}(s) > 1 \)):

\[ \frac{1}{2} \int_0^1 (t^{(s-1)/2} - t^{s/2}) \frac{dt}{t} = \frac{1}{s-1} - \frac{1}{s}. \]

For the first integral on the right hand side we use the substitution \( \tilde{t} = 1/t \) and obtain

\[ \int_0^1 t^{(s-1)/2} \psi\left(\frac{1}{t}\right) \frac{dt}{t} = \int_1^\infty t^{(1-s)/2} \psi(t) \frac{dt}{t}. \]

Putting everything together we get

\[ \xi(s) = \int_0^\infty t^{s/2} \psi(t) \frac{dt}{t} = \int_1^\infty (t^{(1-s)/2} + t^{s/2}) \psi(t) \frac{dt}{t} + \left(\frac{1}{s-1} - \frac{1}{s}\right). \]

The integral on the right hand side converges for all \( s \in \mathbb{C} \) to a holomorphic function in \( \mathbb{C} \). Thus we have got a representation of the function \( \xi(s) \) valid in the whole plane. This representation is invariant under the map \( s \mapsto 1 - s \), proving \( \xi(1 - s) = \xi(s) \), i.e. part a) of the theorem.

To prove part b), we use the equation we just proved:

\[ \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1 - s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \]

yielding

\[ \zeta(1 - s) = \pi^{1/2-s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right)^{-1} \zeta(s). \]

By theorem 9.5.a) we have

\[ \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) = \frac{\pi}{\sin(\pi \frac{1+s}{2})} = \frac{\pi}{\cos(\pi \frac{s}{2})}, \]
therefore
\[ \zeta(1-s) = \pi^{-1/2-s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) \cos\left(\frac{\pi s}{2}\right) \zeta(s). \]

Now by theorem 9.5.b)
\[ \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) = 2^{1-s} \sqrt{\pi} \Gamma(s), \]
which implies
\[ \zeta(1-s) = 2^{1-s} \pi^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s), \quad \text{q.e.d.} \]

10.5. Corollary.  a) For every integer \( k > 0 \)
\[ \zeta(-2k) = 0. \]
These are the only zeroes of the zeta function in the halfplane \( \text{Re}(s) < 0. \)
b) \[ \zeta(0) = -\frac{1}{2}. \]
c) For every integer \( k > 0 \)
\[ \zeta(1-2k) = -\frac{B_{2k}}{2k}. \]

Proof. a) We use the functional equation
\[ \zeta(1-s) = 2^{1-s} \pi^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \zeta(s) \]
\( \text{Re}(1-s) < 0 \) is equivalent to \( \text{Re}(s) > 1. \) Since \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 1 \) (theorem 4.5), the only zeroes of the right hand side for \( \text{Re}(s) > 1 \) come from the cosine function. Now
\[ \cos\left(\frac{\pi s}{2}\right) = 0 \iff s = 1 + 2k \quad \text{with} \ k \in \mathbb{Z} \]
This implies assertion a)
c) From the functional equation we get
\[ \zeta(1-2k) = 2^{1-2k} \pi^{-2k} \Gamma(2k) \cos(\pi k) \zeta(2k) = \frac{2}{(2\pi)^{2k}} (2k-1)! (-1)^k \zeta(2k). \]
By theorem 5.8.ii)
\[ \zeta(2k) = (-1)^k \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}. \]
Substituting this in the equation above yields
\[ \zeta(1-2k) = -\frac{B_{2k}}{2k}. \]

b) We write the functional equation in the form \( \zeta(1-s) = f_1(s)f_2(s) \) with
\[ f_1(s) := 2^{1-s}\pi^{-s}\Gamma(s) \quad \text{and} \quad f_2(s) := \cos\frac{\pi s}{2}\zeta(s). \]

\( f_1 \) is holomorphic in a neighborhood of \( s = 1 \) and \( f_1(1) = 1/\pi \). The function \( f_2 \) is likewise holomorphic in a neighborhood of \( s = 1 \), since the pole of the zeta function is cancelled by the zero of the cosine. To calculate \( f_2(1) \), we determine the first terms of the Taylor resp. Laurent expansions of the factors.

\[ \cos\frac{\pi s}{2} = \cos\left(\frac{\pi}{2}(s-1) + \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}(s-1)\right) = -\frac{\pi}{2}(s-1) + O((s-1)^3), \]
\[ \zeta(s) = \frac{1}{s-1} + (\text{holomorphic function}). \]

Multiplying both expressions yields \( f_2(s) = -\frac{\pi}{2} + O(s-1) \), hence \( f_2(1) = -\frac{\pi}{2} \). Therefore
\[ \zeta(0) = f_1(1)f_2(1) = -\frac{1}{\pi} \quad \text{q.e.d.} \]

10.6. Theorem. For all \( t \in \mathbb{R} \)
\[ \zeta(1+it) \neq 0. \]

Proof. We use the inequality
\[ 3 + 4\cos t + \cos 2t \geq 0 \quad \text{for all} \quad t \in \mathbb{R}. \]

This is proved as follows: Since \( \cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 \), we have
\[ 3 + 4\cos t + \cos 2t = 2(1 + 2\cos t + \cos^2 t) = 2(1 + \cos t)^2 \geq 0. \]

Let now \( s = \sigma + it \) be a complex number with \( \text{Re}(s) = \sigma > 1 \). Then
\[ \log \zeta(s) = \sum_{p \in \mathbb{P}} \log \frac{1}{1-p^{-s}} = \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{p^{ks}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \]
where
\[ a_n = \begin{cases} 1/k, & \text{if } n = p^k \text{ for some prime } p, \\ 0, & \text{otherwise}. \end{cases} \]

Since \( \log |z| = \text{Re}(\log z) \) for every \( z \in \mathbb{C}^* \),
\[ \log |\zeta(s)| = \sum_{n=1}^{\infty} a_n \text{Re}(n^{-s}) = \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \cos(t \log n). \]
Using a trick of v. Mangoldt (1895) we form the expression
\[
\log \left( |\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \right) = \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \left( 3 + 4 \cos(t \log n) + \cos(2t \log n) \right) \geq 0.
\]
Therefore
\[
|\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 1 \quad \text{for all } \sigma > 1 \text{ and } t \in \mathbb{R}.
\]
Assume that \( \zeta(1 + it) = 0 \) for some \( t \neq 0 \). Then the function \( s \mapsto \zeta(s)^3 \zeta(s + it)^4 \) has a zero at \( s = 1 \), since the pole of order 3 of the function \( \zeta(s)^3 \) is compensated by the zero of order \( \geq 4 \) of the function \( \zeta(s + it)^4 \). Therefore
\[
\lim_{\sigma \searrow 1} |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| = 0,
\]
contradicting the above estimate. Hence the assumption is false, which proves the theorem.

10.7. Riemann Hypothesis. It follows from theorem 10.6 and the functional equation that \( \zeta(s) \neq 0 \) for all \( s \in \mathbb{C} \) with \( \Re(s) = 0 \). Therefore, besides the trivial zeroes of the zeta function at \( s = -2k, k \in \mathbb{N}_1 \), all other zeroes of the zeta function must satisfy \( 0 < \Re(s) < 1 \). It was conjectured by Riemann in 1859 that all non-trivial zeroes of the zeta function actually have \( \Re(s) = \frac{1}{2} \). This is the famous Riemann hypothesis which is still unproven today.