6. Dirichlet Series

6.1. Definition. A Dirichlet series is a series of the form

\[ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad (s \in \mathbb{C}), \]

where \((a_n)_{n \geq 1}\) is an arbitrary sequence of complex numbers.

The abscissa of absolute convergence of this series is defined as

\[ \sigma_a := \sigma_a(f) := \inf \{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} < \infty \} \in \mathbb{R} \cup \{\pm\infty\}. \]

If \(\sum_{n=1}^{\infty} (|a_n|/n^\sigma)\) does not converge for any \(\sigma \in \mathbb{R}\), then \(\sigma_a = +\infty\), if it converges for all \(\sigma \in \mathbb{R}\), then \(\sigma_a = -\infty\).

An analogous argument as in the case of the zeta function shows that a Dirichlet series with abscissa of absolute convergence \(\sigma_a\) converges absolutely and uniformly in every halfplane \(\mathcal{H}(\sigma), \sigma > \sigma_a\).

Example. The Dirichlet series

\[ g(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \]

has \(\sigma_a(g) = 1\). We will see however that the series converges for every \(s \in \mathcal{H}(0)\). Of course the convergence is only conditional and not absolute if \(0 < \text{Re}(s) \leq 1\).

We need some preparations.

6.2. Lemma (Abel summation). Let \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\) be two sequences of complex numbers and set

\[ A_n := \sum_{k=1}^{n} a_k, \quad A_0 = 0 \text{ (empty sum)}. \]

Then we have for all \(n \geq m \geq 1\)

\[ \sum_{k=m}^{n} a_kb_k = A_nb_n - A_{m-1}b_m - \sum_{k=m}^{n-1} A_k(b_{k+1} - b_k). \]

Remark. This can be viewed as an analogon of the formula for partial integration

\[ \int_{a}^{b} F'(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_{a}^{b} F(x)g'(x)dx. \]
6. Dirichlet series

Proof.
\[ \sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} (A_k - A_{k-1}) b_k = \sum_{k=m}^{n} A_k b_k - \sum_{k=m-1}^{n-1} A_k b_{k+1} \]
\[ = A_n b_n + \sum_{k=m}^{n-1} A_k b_k - \sum_{k=m}^{n-1} A_k b_{k+1} - A_{m-1} b_m \]
\[ = A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k), \quad \text{q.e.d.} \]

6.3. Lemma. Let \( s \in \mathbb{C} \) with \( \sigma := \text{Re}(s) > 0 \). Then we have for all \( m, n \geq 1 \)
\[ \left| \frac{1}{n^s} - \frac{1}{m^s} \right| \leq \left| s \right| \cdot \left| \frac{1}{n^\sigma} - \frac{1}{m^\sigma} \right|. \]

Proof. We may assume \( n \geq m \). Since \( \frac{d}{dx} \left( \frac{1}{x^s} \right) = -s \cdot \frac{1}{x^{s+1}}, \)
\[ -s \int_{m}^{n} \frac{dx}{x^{s+1}} = \frac{1}{n^s} - \frac{1}{m^s}. \]
Taking the absolute values, we get the estimate
\[ \left| \frac{1}{n^s} - \frac{1}{m^s} \right| \leq \left| s \right| \int_{m}^{n} \frac{dx}{x^{s+1}} = \left| s \right| \cdot \left| \frac{1}{n^\sigma} - \frac{1}{m^\sigma} \right|, \quad \text{q.e.d.} \]

Remark. For \( s_0 \in \mathbb{C} \) and an angle \( \alpha \) with \( 0 < \alpha < \pi/2 \), we define the angular region
\[ \text{Ang}(s_0, \alpha) := \{ s_0 + re^{i\varphi} : r \geq 0 \text{ and } |\varphi| \leq \alpha \}. \]
For any \( s \in \text{Ang}(s_0, \alpha) \setminus \{ s_0 \} \) we have
\[ \frac{|s - s_0|}{\text{Re}(s - s_0)} = \frac{1}{\cos \varphi} \leq \frac{1}{\cos \alpha}, \]
hence the estimate in lemma 6.3 can be rewritten as
\[ \left| \frac{1}{n^s} - \frac{1}{m^s} \right| \leq \frac{1}{\cos \alpha} \cdot \left| \frac{1}{n^\sigma} - \frac{1}{m^\sigma} \right| \quad \text{for all } s \in \text{Ang}(0, \alpha). \]

6.4. Theorem. Let
\[ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \]
be a Dirichlet series such that for some \( s_0 \in \mathbb{C} \) the partial sums \( \sum_{n=1}^{N} \frac{a_n}{n^{s_0}} \) are bounded for \( N \to \infty \). Then the Dirichlet series converges for every \( s \in \mathbb{C} \) with \( \Re(s) > \sigma_0 := \Re(s_0) \).

The convergence is uniform on every compact subset

\[ K \subset H(\sigma_0) = \{ s \in \mathbb{C} : \Re(s) > \sigma_0 \} . \]

Hence \( f \) is a holomorphic function in \( H(\sigma_0) \).

**Proof.** Since

\[
  f(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s_0}} \cdot \frac{a_n}{n^{s-s_0}} = \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n^{s-s_0}} \quad \text{where} \quad \tilde{a}_n := \frac{a_n}{n^{s_0}},
\]

we may suppose without loss of generality that \( s_0 = 0 \). By hypothesis there exists a constant \( C_1 > 0 \) such that

\[
  \left| \sum_{n=1}^{N} a_n \right| \leq C_1 \quad \text{for all} \quad N \in \mathbb{N}.
\]

The compact set \( K \) is contained in some angular region \( \text{Ang}(0, \alpha) \) with \( 0 < \alpha < \pi/2 \). We define

\[
  C_\alpha := \frac{1}{\cos \alpha} \quad \text{and} \quad \sigma_* := \inf\{ \Re(s) : s \in K \} > 0.
\]

Now we apply the Abel summation lemma 6.2 to the sum \( \sum a_n \cdot (1/n^s) \), \( s \in K \). Setting \( A_N := \sum_{n=1}^{N} a_n \), we get for \( N \geq M \geq 1 \)

\[
  \sum_{n=M}^{N} \frac{a_n}{n^s} = A_N \frac{1}{N^s} - A_{M-1} \frac{1}{M^s} + \sum_{n=M}^{N-1} A_n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right).
\]

This leads to the estimate (with \( \sigma = \Re(s) \))

\[
  \left| \sum_{n=M}^{N} \frac{a_n}{n^s} \right| \leq 2C_1 \left| \frac{1}{M^s} \right| + C_1 \sum_{n=M}^{N-1} \left| \frac{1}{n^s} - \frac{1}{(n+1)^s} \right|
  \leq 2C_1 \frac{1}{M^\sigma} + C_1 C_\alpha \sum_{n=M}^{N-1} \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right)
  = 2C_1 \frac{1}{M^\sigma} + C_1 C_\alpha \left( \frac{1}{M^\sigma} - \frac{1}{N^\sigma} \right)
  \leq \frac{C_1}{M^\sigma} (2 + C_\alpha) \leq \frac{C_1 (2 + C_\alpha)}{M^{\sigma_*}}.
\]

6.3
Dirichlet series

This becomes arbitrarily small if \( M \) is sufficiently large. This implies the asserted uniform convergence on \( K \) of the Dirichlet series.

6.5. Theorem. Let

\[
f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

be a Dirichlet series which converges for some \( s_0 \in \mathbb{C} \). Then the series converges uniformly in every angular region \( \text{Ang}(s_0, \alpha) \), \( 0 < \alpha < \pi/2 \). In particular

\[
limit_{s \to s_0} f(s) = f(s_0),
\]

when \( s \) approaches \( s_0 \) within an angular region \( \text{Ang}(s_0, \alpha) \).

Proof. As in the proof of theorem 6.4 we may suppose \( s_0 = 0 \). Set \( C_\alpha := 1/ \cos \alpha \). Let \( \varepsilon > 0 \) be given. Since \( \sum_{n=1}^{\infty} a_n \) converges, there exists an \( n_0 \in \mathbb{N} \), such that

\[
\left| \sum_{n=M}^{N} a_n \right| < \varepsilon_1 := \frac{\varepsilon}{1 + C_\alpha} \quad \text{for all } N \geq M \geq n_0.
\]

With \( A_{Mn} := \sum_{k=M}^{n} a_k \), \( A_{M,M-1} = 0 \), we have by the Abel summation formula

\[
\sum_{n=M}^{N} \frac{a_n}{n^s} = A_{MN} \frac{1}{N^s} + \sum_{n=M}^{N-1} A_{Mn} \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right).
\]

From this, we get for all \( s \in \text{Ang}(0, \alpha) \), \( \sigma := \text{Re}(s) \), and \( N \geq M \geq n_0 \) the estimate

\[
\left| \sum_{n=M}^{N} \frac{a_n}{n^s} \right| \leq \varepsilon_1 \frac{1}{|N^s|} + \varepsilon_1 \sum_{n=M}^{N-1} \left| \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right|
\]

\[
\leq \varepsilon_1 + \varepsilon_1 C_\alpha \sum_{n=M}^{N-1} \left( \frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right)
\]

\[
= \varepsilon_1 + \varepsilon_1 C_\alpha \left( \frac{1}{M^\sigma} - \frac{1}{N^\sigma} \right) \leq \varepsilon_1 + \varepsilon_1 C_\alpha = \varepsilon.
\]

This shows the uniform convergence of the Dirichlet series in \( \text{Ang}(0, \alpha) \). Therefore \( f \) is continuous in \( \text{Ang}(0, \alpha) \), which implies the last assertion of the theorem.

6.6. Definition. Let \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) be a Dirichlet series. The abscissa of convergence of \( f \) is defined by

\[
\sigma_c := \sigma_c(f) := \inf \{ \text{Re}(s) : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges} \}.
\]
By theorem 6.4 this is the same as
\[ \sigma_c = \inf \{ \Re(s) : \sum_{n=1}^{N} a_n n^s \text{ is bounded for } N \to \infty \} \]
and it follows that the series converges to a holomorphic function in the halfplane \( H(\sigma_c) \).

**Examples.** Consider the three Dirichlet series
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad g(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}. \]

We have \( \sigma_c(\zeta) = \sigma_c(g) = \sigma_c(1/\zeta) = 1 \). Clearly \( \sigma_c(\zeta) = 1 \) and \( \sigma_c(g) = 0 \), since the partial sums \( \sum_{n=1}^{N} (-1)^{n-1} \) are bounded. The abscissa of convergence \( \sigma_c(1/\zeta) \) is not known; of course \( \sigma_c(1/\zeta) \leq 1 \). One conjectures that \( \sigma_c(1/\zeta) = \frac{1}{2} \), which is equivalent to the Riemann Hypothesis, which we will discuss in a later chapter.

**Remark.** Multiplying the zeta series by \( 2^{-s} \) yields \( 2^{-s} \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \). Hence
\[ g(s) = (1 - 2^{1-s}) \zeta(s). \]

**6.7. Theorem.** If the Dirichlet series \( f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \) has a finite abscissa of convergence \( \sigma_c \), then for the abscissa of absolute convergence \( \sigma_a \) the following estimate holds:
\[ \sigma_c \leq \sigma_a \leq \sigma_c + 1. \]

**Proof.** Without loss of generality we may suppose \( \sigma_c = 0 \). Then \( \sum_{n=1}^{\infty} \frac{a_n}{n^\varepsilon} \) converges for every \( \varepsilon > 0 \). We have to show that
\[ \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_s}} < \infty \quad \text{for all } \sigma_s > 1. \]
To see this, write \( \sigma_s = 1 + 2\varepsilon, \varepsilon > 0 \). Then
\[ \frac{|a_n|}{n^{\sigma_s}} = \frac{|a_n|}{n^\varepsilon} \cdot \frac{1}{n^{1+\varepsilon}}. \]
Since \( |a_n|/n^\varepsilon \) is bounded for \( n \to \infty \) and \( \sum_{n=1}^{\infty} 1/n^{1+\varepsilon} < \infty \), the assertion follows.

**Remarks.** a) It can be easily seen that \( \sigma_c = -\infty \) implies \( \sigma_a = -\infty \).

b) The above examples show that the cases \( \sigma_a = \sigma_c \) and \( \sigma_a = \sigma_c + 1 \) do actually occur.
c) That $\sigma_a$ and $\sigma_\infty$ may be different is quite surprising if one looks at the situation for power series: If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$, it converges absolutely for every $z$ with $|z| < |z_0|$.

6.8. Theorem (Landau). Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with non-negative coefficients $a_n \geq 0$ and finite abscissa of absolute convergence $\sigma_a \in \mathbb{R}$. Then the function $f$, which is holomorphic in the halfplane $H(\sigma_a)$, cannot be continued analytically as a holomorphic function to any neighborhood of $\sigma_a$.

Proof. Assume to the contrary that there exists a small open disk $D$ around $\sigma_a$ such that $f$ can be analytically continued to a holomorphic function in $H(\sigma_a) \cup D$, which we denote again by $f$. Then the Taylor series of $f$ at the point $\sigma_1 := \sigma_a + 1$ has radius of convergence $> 1$. Since

$$f^{(k)}(\sigma_1) = \sum_{n=1}^{\infty} \frac{(-\log n)^k a_n}{n^{\sigma_1}},$$

the Taylor series has the form

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\sigma_1)}{k!} (s - \sigma_1)^k = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-\log n)^k a_n}{k! n^{\sigma_1}} (s - \sigma_1)^k.$$

By hypothesis there exists a real $\sigma < \sigma_a$ such that the Taylor series converges for $s = \sigma$. We have

$$f(\sigma) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(\log n)^k a_n (\sigma_1 - \sigma)^k}{k! n^{\sigma_1}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(\log n)^k a_n (\sigma_1 - \sigma)^k}{k!} \cdot \frac{a_n}{n^{\sigma_1}},$$

where the reordering is allowed since all terms are non-negative. Now

$$\sum_{k=0}^{\infty} \frac{(\log n)^k (\sigma_1 - \sigma)^k}{k!} = e^{(\log n)(\sigma_1 - \sigma)} = \frac{1}{n^{\sigma - \sigma_1}},$$

hence we have a convergent series

$$f(\sigma) = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma_1}} \cdot \frac{a_n}{n^{\sigma_1}} = \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma_1}}.$$

Thus the abscissa of absolute convergence is $\leq \sigma < \sigma_a$, a contradiction. Hence the assumption is false, which proves the theorem.
6.9. **Theorem** (Identity theorem for Dirichlet series). Let

\[ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \]

be two Dirichlet series that converge in a common halfplane \( H(\sigma_0) \). If there exists a sequence \( s_\nu \in H(\sigma_0), \nu \in \mathbb{N}_1, \) with \( \lim_{\nu \to \infty} \text{Re}(s_\nu) = \infty \) and

\[ f(s_\nu) = g(s_\nu) \quad \text{for all} \ \nu \geq 1, \]

then \( a_n = b_n \) for all \( n \geq 1 \).

**Proof.** Passing to the difference \( f - g \) shows that it suffices to prove the theorem for the case where \( g \) is identically zero. So we suppose that \( f(s_\nu) = 0 \) for all \( \nu \geq 1 \).

If not all \( a_n = 0 \), then there exists a minimal \( k \) such that \( a_k \neq 0 \). We have

\[ f(s) = \frac{1}{k^s} \left( a_k + \sum_{n > k} \frac{a_n}{(n/k)^s} \right). \]

It suffices to show that there exists a \( \sigma_\star \in \mathbb{R} \) such that

\[ \left| \sum_{n > k} \frac{a_n}{(n/k)^s} \right| \leq \frac{|a_k|}{2} \quad \text{for all} \ s \ \text{with} \ \text{Re}(s) \geq \sigma_\star, \]

for this would imply \( f(s) \neq 0 \) for \( \text{Re}(s) \geq \sigma_\star \), contradicting \( f(s_\nu) = 0 \) for all \( \nu \). The sum \( \sum_{n > k} \frac{a_n}{(n/k)^s} \) converges absolutely for some \( \sigma' \in \mathbb{R} \). Therefore we can find an \( M \geq k \) such that

\[ \sum_{n > M} \frac{|a_n|}{(n/k)^s} \leq \frac{|a_k|}{4}. \]

Further there exists a \( \sigma'' \in \mathbb{R} \) such that

\[ \sum_{k < n \leq M} \frac{|a_n|}{(n/k)^s} \leq \frac{|a_k|}{4}. \]

Combining the last two estimates shows

\[ \left| \sum_{n > k} \frac{a_n}{(n/k)^s} \right| \leq \frac{|a_k|}{2} \quad \text{for all} \ s \ \text{with} \ \text{Re}(s) \geq \max(\sigma', \sigma''), \quad \text{q.e.d.} \]

**Remark.** A similar theorem is not true for arbitrary holomorphic functions in halfplanes. For example, the sine function satisfies

\[ \sin(\pi n) = 0 \quad \text{for all integers} \ n, \]
without being identically zero. This shows also that not every function holomorphic in 

a halfplane \( H(\sigma) \) can be expanded in a Dirichlet series.

**6.10. Theorem.** Let \( a, b : \mathbb{N}_1 \to \mathbb{C} \) be two arithmetical functions such that the Dirichlet series

\[
f(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \text{and} \quad g(s) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s}
\]

converge absolutely in a common halfplane \( H(\sigma_0) \). Then we have for the product

\[
f(s)g(s) = \sum_{n=1}^{\infty} \frac{(a \ast b)(n)}{n^s}.
\]

This Dirichlet series converges absolutely in \( H(\sigma_0) \).

**Proof.** Since the series for \( f(s) \) and \( g(s) \) converge absolutely for \( s \in H(\sigma_0) \), they can be multiplied term by term

\[
f(s)g(s) = \sum_{k=1}^{\infty} \frac{a(k)}{k^s} \sum_{\ell=1}^{\infty} \frac{b(\ell)}{\ell^s} = \sum_{k, \ell \geq 1} a(k)b(\ell) \frac{1}{k^s\ell^s}
\]

\[
= \sum_{n=1}^{\infty} \sum_{k\ell=n} a(k)b(\ell) \frac{1}{(k\ell)^s} = \sum_{n=1}^{\infty} \frac{(a \ast b)(n)}{n^s},
\]

and the product series converges absolutely, q.e.d.

**Examples.** i) The zeta function \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) is the Dirichlet series associated to the constant arithmetical function \( u(n) = 1 \). Since \( u \ast \mu = \delta_1 \), it follows

\[
(\sum_{n=1}^{\infty} \frac{1}{n^s})(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}) = \sum_{n=1}^{\infty} \frac{\delta_1(n)}{n^s} = 1,
\]

which gives a new proof of

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \text{(cf. theorem 4.5)}.
\]

ii) The Dirichlet series associated to the identity map \( \iota : \mathbb{N}_1 \to \mathbb{N}_1 \) is

\[
\sum_{n=1}^{\infty} \frac{n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} = \zeta(s-1),
\]
which converges absolutely for \( \text{Re}(s) > 2 \). For the divisor sum function \( \sigma \) we have \( u \ast \iota = \sigma \), cf. (3.15.iii), which implies

\[
\zeta(s)\zeta(s - 1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \quad \text{for } \text{Re}(s) > 2.
\]

iii) In a similar way, the formula \( \varphi = \mu \ast \iota \) for the Euler phi function, cf. (3.15.i), yields

\[
\frac{\zeta(s - 1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} \quad \text{for } \text{Re}(s) > 2.
\]

6.11. Theorem (Euler product for Dirichlet series). Let \( a : \mathbb{N}_1 \to \mathbb{C} \) be a multiplicative arithmetical function such that the Dirichlet series

\[
f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}
\]

has abscissa of absolute convergence \( \sigma_a < \infty \).

a) Then we have in \( H(\sigma_a) \) the product representation

\[
f(s) = \prod_{p \in \mathbb{P}} \left( \sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}} \right) = \prod_{p \in \mathbb{P}} \left( 1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \frac{a(p^3)}{p^{3s}} + \cdots \right),
\]

where the product is extended over the set \( \mathbb{P} \) of all primes.

b) If the arithmetical function \( a \) is completely multiplicative, this can be simplified to

\[
f(s) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{a(p)}{p^s} \right)^{-1}.
\]

Proof. Let \( \mathcal{P} \subset \mathbb{P} \) be a finite set of primes and \( \mathbb{N}(\mathcal{P}) \) the set of all positive integers whose prime decomposition contains only primes from the set \( \mathcal{P} \). Since \( a \) is multiplicative, we have for an integer \( n \) with prime decomposition \( n = p_1^{k_1}p_2^{k_2} \cdots p_r^{k_r} \)

\[
a(n) = a(p_1^{k_1})a(p_2^{k_2}) \cdots a(p_r^{k_r}).
\]

It follows by multiplying the infinite series term by term that

\[
\prod_{p \in \mathcal{P}} \left( 1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \frac{a(p^3)}{p^{3s}} + \cdots \right) = \sum_{n \in \mathbb{N}(\mathcal{P})} \frac{a(n)}{n^s}.
\]

Letting \( \mathcal{P} = \mathcal{P}_m \) be set of all primes \( \leq m \) and passing to the limit \( m \to \infty \), we obtain part a) the theorem.
If $a$ is completely multiplicative, then $a(p^k) = a(p)^k$, hence
\[
\sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}} = \sum_{k=0}^{\infty} \left( \frac{a(p)}{p^s} \right)^k = \left( 1 - \frac{a(p)}{p^s} \right)^{-1},
\]
proving part b).

**Examples.**

i) The Euler product for the zeta function
\[
\zeta(s) = \prod_{p \in \mathbb{P}} \left( 1 - \frac{1}{p^s} \right)^{-1}
\]
is a special case of this theorem.

ii) Since $\mu(p) = -1$ and $\mu(p^k) = 0$ for $k \geq 2$, the formula for the inverse of the zeta function
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p \in \mathbb{P}} \left( 1 - \frac{1}{p^s} \right) = \frac{1}{\zeta(s)}
\]
also follows from this theorem.