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# Analytic Number Theory

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at the Department of Mathematics, LMU Munich, Germany

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# 0. Notations and Conventions

## Standard notations for sets

$\mathbb{Z}$ ring of all integers

$\mathbb{N}_0$ set of all integers $\geq 0$

$\mathbb{N}_1$ set of all integers $\geq 1$

$\mathbb{P}$ set of all primes $= \{2, 3, 5, 7, 11, \ldots\}$

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the fields of rational, real and complex numbers respectively

$A^*$ multiplicative group of invertible elements of a ring $A$

$[a, b], ]a, b[\,, [a, b[\,, ]a, b]$ denote closed, open and half-open intervals of $\mathbb{R}$

$\mathbb{R}_+ = [0, \infty[$ set of non-negative real numbers

$\mathbb{R}_+^* = \mathbb{R}_+ \cap \mathbb{R}^*$ multiplicative group of positive real numbers

$\lfloor x \rfloor$ greatest integer $\leq x \in \mathbb{R}$

## Landau symbols *O*, *o*

For two functions $f, g : [a, \infty[ \to \mathbb{C}$, one writes

$$f(x) = O(g(x)) \quad \text{for } x \to \infty,$$

if there exist constants $C > 0$ and $x_0 \geq a$ such that

$$|f(x)| \leq C|g(x)| \quad \text{for all } x \geq x_0.$$

Similarly,

$$f(x) = o(g(x)) \quad \text{for } x \to \infty$$

means that for every $\varepsilon > 0$ there exists $R \geq a$ such that

$$|f(x)| \leq \varepsilon|g(x)| \quad \text{for all } x \geq R.$$

For functions $f, g : ]a, b[ \to \mathbb{C}$ the notions

$$f(x) = O(g(x)) \quad \text{for } x \searrow a,$$

and

$$f(x) = o(g(x)) \quad \text{for } x \searrow a,$$

are defined analogously.

$$f(x) = f_0(x) + O(g(x))$$

is defined as $f(x) - f_0(x) = O(g(x))$.

### Asymptotic equality

Two functions $f, g : [a, \infty[ \to \mathbb{C}$ are said to be asymptotically equal for $x \to \infty$, in symbols

$$f(x) \sim g(x) \quad \text{for } x \to \infty,$$

if $g(x) \neq 0$ for $x \geq x_0$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Analogously, for two sequences $(a_n)_{n \geqslant n_0}$ and $(b_n)_{n \geqslant n_0}$,

$$a_n \sim b_n$$

means $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$. A famous example for asymptotic equality is the Stirling formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

which we will prove in Chap. 9.

### Miscellaneous

We sometimes write ‘iff’ as an abbreviation for ‘if and only if’.