Mathematisches Institut
der Universität München
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SoSe 2024
June 26, 2024

Selected Topics from Number Theory

Problem sheet #7

Problem 25

a) Determine all units of the ring $\mathbb{Z}[\sqrt{2}]$.

b) Determine all units of the ring $\mathbb{Z}[\sqrt{3}]$.

Problem 26

a) Prove that there are infinitely many integers $n \geqslant 1$ such that the sum

$$1 + 2 + \ldots + n = \sum_{k=1}^{n} k$$

is a square.

Hint. Problem 25a) may be useful,

b)* (Unsolved problem: Erdös-Moser equation)
Do there exist solutions of the Diophantine equation

$$1^k + 2^k + \ldots + (n-1)^k = n^k$$

apart from the trivial case $(n, k) = (3, 1)$?

Problem 27 Let $\mathcal{C}^\infty(\mathbb{R}_+^*)$ be the vector space of all infinitely differentiable functions $f : \mathbb{R}_+^* \to \mathbb{R}$. For $p \in \mathbb{R}$, define linear operators

$$S_p, T_p, B_p : \mathcal{C}^\infty(\mathbb{R}_+^*) \longrightarrow \mathcal{C}^\infty(\mathbb{R}_+^*)$$

by

$$\begin{aligned}
(S_p f)(x) &:= f'(x) - \frac{p}{x} f(x), \\
(T_p f)(x) &:= f'(x) + \frac{p}{x} f(x), \\
(B_p f)(x) &:= f''(x) + \frac{1}{x} f'(x) - \Big(1 + \frac{p^2}{x^2}\Big) f(x).
\end{aligned}$$

Remark.

$$B_p y = y'' + \frac{1}{x} y' - \Big(1 + \frac{p^2}{x^2}\Big) y = 0$$

is the *modified Bessel differential equation*, it's solutions are called modified Bessel functions.

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a) Prove the following formulas

i) $T_{p+1}S_pf = B_pf + f,$

ii) $S_{p-1}T_pf = B_pf + f,$

iii) $S_pB_pf = B_{p+1}S_pf,$

iv) $T_pB_pf = B_{p-1}T_pf.$

b) Let $V_p := \{f \in \mathcal{C}^\infty(\mathbb{R}_+^*) : B_pf = 0\}$.
Prove that S_p resp T_{p+1} define isomorphisms

$$S_p : V_p \longrightarrow V_{p+1}, \qquad T_{p+1} : V_{p+1} \longrightarrow V_p,$$

which are inverses of each other.

Problem 28 (Continuation of Problem 27) If p is not a negative integer, define

$$I_p(x) := \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2k}$$

a) Prove

$$(S_pI_p)(x) = I_{p+1}(x), \quad \text{and} \quad (T_pI_p)(x) = I_{p-1}(x)$$

and concude that $B_pI_p = 0$, i.e. I_p is a modified Bessel function.

b) Prove the formulas

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x \quad \text{and} \quad I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$$

These problems will be discussed Wednesday, July 10, 2024