Mathematisches Institut
der Universität München
Prof. Otto Forster

SoSe 2024
June 19, 2024

Selected Topics from Number Theory

Problem sheet #6

Problem 21 Let f_n be the Fibonacci numbers. Prove

a) $$\det\begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix} = (-1)^m \quad \text{for all } m \geqslant 1.$$

b) $$\det\begin{pmatrix} f_{m+2} & f_m \\ f_m & f_{m-2} \end{pmatrix} = (-1)^{m+1} \quad \text{for all } m \geqslant 2.$$

c) If $m+k$ is even and $m \geqslant k$, then
$$f_{m+k}f_{m-k} = (f_m + f_k)(f_m - f_k).$$

d) What about the case $m+k$ odd?

Problem 22 Let d be a positive squarefree integer.
Suppose that $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ is a solution of
$$x^2 - dy^2 = -4.$$
Construct a solution $(u,v) \in \mathbb{Z} \times \mathbb{Z}$ of
$$u^2 - dv^2 = -1.$$
Hint. If x is odd, calculate the element $(x + y\sqrt{d})^3 \in \mathbb{Z}[\sqrt{d}]$.

Problem 23 Let R be a commutative ring with unit element and $D \in R$. Define
$$M(D,R) := \left\{ A = \begin{pmatrix} x & yD \\ y & x \end{pmatrix} : \ x, y \in R \right\}.$$

a) Prove that $M(D,R)$ is a commutative ring with respect to matrix addition and multiplication, which can be viewed as an extension ring of R via the embedding
$$R \longrightarrow M(D,R), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

b) Show that in the special case $R = \mathbb{R}$ and $D = -1$, the ring $M(-1,\mathbb{R})$ is isomorphic to the field $\mathbb{C}$ of complex numbers.

---

p.t.o.

Problem 24 (Continuation of Problem 23)

a) Show that

$$G(D, R) := \{A \in M(D, R) : \det A = 1\}$$

is a multiplicative commutative group.

b) In the special case $R := \mathbb{R}$ and $D > 0$ define

$$G_+(D, \mathbb{R}) := \left\{ A = \begin{pmatrix} x & yD \\ y & x \end{pmatrix} \in G(D, \mathbb{R}) : x > 0 \right\}.$$

Prove that $G_+(D, \mathbb{R})$ is a subgroup of $G(D, \mathbb{R})$.

c) Does the analogue of b) remain true, if $\mathbb{R}$ is replaced by an arbitrary ordered field K ?

These problems will be discussed Wednesday, July 3, 2024