Mathematisches Institut
der Universität München
Prof. Otto Forster

SoSe 2024
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Selected Topics from Number Theory

Problem sheet #4

Problem 13 By definition, $\mathrm{SL}(2,\mathbb{Z})$ is the group of all integer 2×2-matrices with determinant 1.

Define

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_- := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Prove that these matrices generate $\mathrm{SL}(2,\mathbb{Z})$ in the following sense:
Every $X \in \mathrm{SL}(2,\mathbb{Z})$ can be written as a finite product

$$X = M_1 \cdot M_2 \cdot \ldots \cdot M_r \quad \text{with } M_j \in \{T, T_-, S\}, \ 1 \leqslant j \leqslant r.$$

Problem 14 (Continuation of Problem 13)

Let Γ be the group of all integer 2×2-matrices with determinant ± 1.

Prove that Γ is generated by the matrices

$$T, \ T_- \text{ and } R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 15 Prove the following identity (detected by D. P. Dalzell 1944)

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi$$

Problem 16 (Continuation of Problem 15) Using Euler's Beta Function

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt = B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

estimate the difference $22/7 - \pi$ and compare it with Archimedes' estimate

$$3\frac{1}{7} > \pi > 3\frac{10}{71}$$

and the estimate obtained by the continued fraction expansion

$$\pi = \mathrm{cfrac}(3, 7, 15, \ldots).$$

These problems will be discussed Wednesday, June 5, 2024, 16-18 h