Mathematisches Institut
der Universität München
Prof. Otto Forster

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Selected Topics from Number Theory

Problem sheet #3, Solutions

Problem 9 The sequence $(f_\nu)_{\nu\geqslant 0}$ of Fibonacci numbers is recursively defined by

$$f_0 := 0,\ \ f_1 := 1, \qquad f_{n+1} := f_n + f_{n-1}, \quad (n \geqslant 1).$$

Show that the limit $\lim\limits_{n\to\infty} \dfrac{f_{n+1}}{f_n}$ exists and equals the *golden ratio*

$$\phi := \frac{1+\sqrt{5}}{2}.$$

Solution. To get a solution of the recurrence relation $f_{n+1} := f_n + f_{n-1}$, we try an 'Ansatz'

$$f_n = \lambda^n.$$

The recurrence relation will be satisfied for all n if

$$\lambda^2 + \lambda + 1 = 0 \qquad \Longleftrightarrow \qquad \lambda = \lambda_{1/2} = \frac{1 \pm \sqrt{5}}{2},$$

i.e.

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \phi \approx 1.618\ldots, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} = 1 - \phi = -\phi^{-1}.$$

Of course also every linear combination $c_1\lambda_1^n + c_2\lambda_2^n$ will satisfy the recurrence relation. To meet the initial conditions, we have to find coefficients $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned} &f_0 = 0 = c_1\lambda_1^0 + c_2\lambda_2^0 = c_1 + c_2 \\ &f_1 = 1 = c_1\lambda_1 + c_2\lambda_2. \end{aligned}$$

One finds $c_1 = -c_2 = \dfrac{1}{\sqrt{5}}$, therefore

$$f_n = \frac{1}{\sqrt{5}}\left(\phi^n - (-1)^n\phi^{-n}\right) \qquad \text{(Formula of Moivre-Binet)}$$

Now

$$\frac{f_{n+1}}{f_n} = \frac{\phi^{n+1} - (-1)^{n+1}\phi^{-n-1}}{\phi^n - (-1)^n\phi^{-n}} = \phi \cdot \frac{1 - (-1)^{n+1}\phi^{-2n-2}}{1 - (-1)^n\phi^{-2n}}$$

Since $\lim\limits_{k\to\infty} \phi^{-k} = 0$, we get $\lim\limits_{n\to\infty} f_{n+1}/f_n = \phi$, q.e.d.

Problem 10 (Continuation of Problem 9)
a) Show that the CF expansion of ϕ is

$$\phi = \mathrm{cfrac}(1, 1, 1, 1, 1, \ldots)$$

and that the n-th convergent $\dfrac{p_n}{q_n}$ of this continued fraction equals $\dfrac{f_{n+2}}{f_{n+1}}$.
b) Prove that for every constant $c > \sqrt{5}$ the inequality

$$\left|\phi - \frac{p}{q}\right| < \frac{1}{cq^2}, \quad p, q \text{ positive integers},$$

has only a finite number of solutions.

Solution. a) Since $\phi \approx 1.618$, the CF expansion begins with

$$\phi = 1 + \frac{1}{\xi_1},$$

where

$$\xi_1 = \frac{1}{\phi - 1} = \frac{2}{-1 + \sqrt{5}} = \frac{2(1 + \sqrt{5})}{(\sqrt{5})^2 - 1} = \frac{2(1 + \sqrt{5})}{4} = \frac{1 + \sqrt{5}}{2} = \phi$$

So we have already reached periodicity after one step, which implies

$$\phi = \mathrm{cfrac}(1, 1, 1, 1, 1, \ldots)$$

The recurrence relations for the numerator and denominator of the convergents p_n/q_n of the continued are the same as for the Fibonacci numbers

$$\begin{aligned} p_n &= 1 \cdot p_{n-1} + p_{n-2}, \\ q_n &= 1 \cdot q_{n-1} + q_{n-1}. \end{aligned}$$

However the initial conditions are different:

$$\begin{pmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Therefore

$$p_0 = 1 = f_2, \;\; p_1 = 2 = f_3 \qquad \Rightarrow \qquad p_n = f_{n+2}$$

and

$$q_0 = 1 = f_1, \;\; q_1 = 1 = f_2 \qquad \Rightarrow \qquad q_n = f_{n+1}.$$

b) Let $c > \sqrt{5}$. We have to show that

$$\left|\phi - \frac{f_{n+1}}{f_n}\right| < \frac{1}{c} \cdot \frac{1}{f_n^2}$$

is satisfied for at most finitely many $n \in \mathbb{N}$.
Multiplying by f_n^2, this inequality is equivalent to

$$\left|f_n^2\phi - f_n f_{n+1}\right| < \frac{1}{c} \tag{$*$}$$

Now

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - \varepsilon\phi^{-n}), \qquad f_{n+1} = \frac{1}{\sqrt{5}}(\phi^{n+1} + \varepsilon\phi^{-n-1}), \qquad \text{with } \varepsilon = (-1)^n.$$

From this we get

$$\begin{aligned} f_n^2\phi &= \frac{1}{5}\left(\phi^n - \varepsilon\phi^{-n}\right)^2\phi = \frac{1}{5}\left(\phi^{2n+1} - 2\varepsilon\phi + \phi^{-2n+1}\right), \\ f_n f_{n+1} &= \frac{1}{5}\left(\phi^n - \varepsilon\phi^{-n}\right)\left(\phi^{n+1} + \varepsilon\phi^{-n-1}\right) = \frac{1}{5}\left(\phi^{2n+1} - \varepsilon\phi + \varepsilon\phi^{-1} - \phi^{-2n-1}\right) \end{aligned}$$

and, using $\phi + \phi^{-1} = \sqrt{5}$,

$$f_n^2\phi - f_n f_{n+1} = \frac{1}{5}\left(-\varepsilon\phi - \varepsilon\phi^{-1}\right) + O\left(\phi^{-2n+1}\right) = -\frac{(-1)^n}{\sqrt{5}} + O\left(\phi^{-2n+1}\right).$$

But this contradicts $(*)$, if n is sufficienty large.

Problem 11 Sylvester's sequence $(S_n)_{n\geqslant 0}$ is recursively defined by

$$S_0 := 2, \qquad S_n := 1 + \prod_{\nu=0}^{n-1} S_\nu, \quad (n \geqslant 1).$$

Hence the series begins with $(2, 3, 7, 43, 1807, 3263443, \ldots)$.

a) Show that the series can also be defined by

$$S_0 := 2, \qquad S_{n+1} := S_n^2 - S_n + 1, \quad (n \geqslant 0).$$

b) Prove that

$$\sum_{n=0}^{\infty} \frac{1}{S_n} = 1.$$

Hint. Show that for every $m \geqslant 1$ one has

$$1 = \sum_{\nu=0}^{m-1} \frac{1}{S_\nu} + \frac{1}{S_m - 1}.$$

Problem 12 (Continuation of Problem 11)
Cahen's constant is defined by

$$C := \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{S_\nu - 1}$$

a) Show that another way to define C is

$$C := \sum_{\nu=0}^{\infty} \frac{1}{S_{2\nu}}.$$

b) Consider the CF expansion $C = \mathrm{cfrac}(a_0, a_1, a_2, a_3, \ldots)$ and prove that all coefficients a_ν are squares, in fact

$$C = \mathrm{cfrac}(0, 1, 1, 1, 4, 9, 196, 16641, \ldots).$$

Solution.

a) Since $S_{n+1} - 1 = S_n^2 - S_n = S_n(S_n - 1)$, one has

$$\frac{1}{S_n - 1} - \frac{1}{S_{n+1} - 1} = \frac{1}{S_n - 1} - \frac{1}{S_n(S_n - 1)} = \frac{S_n - 1}{S_n(S_n - 1)} = \frac{1}{S_n}.$$

Therefore

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - 1} = \sum_{k=0}^{\infty} \Big(\frac{1}{S_{2k} - 1} - \frac{1}{S_{2k+1} - 1} \Big) = \sum_{k=0}^{\infty} \frac{1}{S_{2k}}, \qquad \text{q.e.d.}$$

b) Define $a_0 := 0$, $a_1 := 1$, $a_2 := 1$, and

$$\begin{aligned} C_0 &:= \mathrm{cfrac}(a_0) = \frac{p_0}{q_0} = \frac{0}{1}, \\ C_1 &:= \mathrm{cfrac}(a_0, a_1) = \frac{p_1}{q_1} = \frac{1}{1}, \\ C_2 &:= \mathrm{cfrac}(a_0, a_1, a_2) = \frac{p_2}{q_2} = \frac{1}{2} \end{aligned}$$

For $k \geqslant 2$ we define by induction

$$a_k := q_{k-2}^2, \quad C_k := \mathrm{cfrac}(a_0, \ldots, a_k) = \frac{p_k}{q_k}.$$

By the recursion formulas for p_k, q_k

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} = q_{k-2}^2 p_{k-1} + p_{k-2}, \\ q_k &= a_k q_{k-1} + q_{k-2} = q_{k-2}^2 q_{k-1} + q_{k-2}. \end{aligned}$$

Now we assert that

$$S_n - 1 = q_n q_{n+1} \qquad \text{for all } n \geqslant 0.$$

For $n = 0$ this is true since $S_0 = 2$ and $q_0 = q_1 = 1$.
Induction step $n \to n + 1$.

$$\begin{aligned} S_{n+1} - 1 &= S_n^2 - S_n = S_n(S_n - 1) \\ &= (q_n q_{n+1} + 1) q_n q_{n+1} \\ &= q_n^2 q_{n+1}^2 + q_n q_{n+1} \\ &= q_{n+1}(q_n^2 q_{n+1} + q_n) \\ &= q_{n+1} q_{n+2}, \qquad \text{q.e.d.} \end{aligned}$$

To complete the proof, we use the general fact

$$\mathrm{cfrac}(a_0, \ldots, a_n) = a_0 + \sum_{k=0}^{n-1} \frac{(-1)^k}{q_k q_{k+1}}.$$

In our case this yields

$$\mathrm{cfrac}(a_0, \ldots, a_n) = \sum_{k=0}^{n-1} \frac{(-1)^k}{S_k - 1}.$$

Passing to the limit $n \to \infty$ we get

$$\mathrm{cfrac}(a_0, a_1, a_2, a_3, \ldots) = \sum_{k=0}^{\infty} \frac{(-1)^k}{S_k - 1} = C,$$

and by definition all a_k are squares.
