

Selected Topics from Number Theory
Problem sheet #2, Solutions

Problem 5 Let $x = \text{cfrac}(a_0, a_1, a_2, a_3, \dots)$ be the CF expansion of an irrational number x and let

$$\frac{p_n}{q_n} = \text{cfrac}(a_0, a_1, a_2, \dots, a_n), \quad n \geq 0$$

be its convergents. Prove

$$\left| x - \frac{p_n}{q_n} \right| > \frac{1}{q_n(q_n + q_{n+1})}.$$

Solution. We use the fact that the convergents $\frac{p_n}{q_n}$ and $\frac{p_{n+2}}{q_{n+2}}$ lie on the same side of x .

Therefore

$$\left| x - \frac{p_n}{q_n} \right| \geq \left| \frac{p_n}{q_n} - \frac{p_{n+2}}{q_{n+2}} \right|.$$

We now estimate the latter difference

$$\begin{aligned} \left| \frac{p_n}{q_n} - \frac{p_{n+2}}{q_{n+2}} \right| &= \left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} + \frac{p_{n+1}}{q_{n+1}} - \frac{p_{n+2}}{q_{n+2}} \right| \\ &= \left| \frac{(-1)^{n+1}}{q_n q_{n+1}} + \frac{(-1)^{n+2}}{q_{n+1} q_{n+2}} \right| \\ &= \frac{1}{q_n q_{n+1}} - \frac{1}{q_{n+1} q_{n+2}} = \frac{q_{n+2} - q_n}{q_n q_{n+1} q_{n+2}} = \frac{a_{n+2} q_{n+1}}{q_n q_{n+1} q_{n+2}} \\ &= \frac{a_{n+2}}{q_n (a_{n+2} q_{n+1} + q_n)} = \frac{1}{q_n (q_{n+1} + q_n / a_{n+2})} \geq \frac{1}{q_n (q_n + q_{n+1})}. \end{aligned}$$

Problem 6 Let $\text{cfrac}(a_0, a_1, a_2, \dots)$ be an infinite continued fraction with $a_\nu \in \mathbb{R}$ (not necessarily integers), $a_\nu > 0$ for $\nu \geq 1$. Prove that the continued fraction converges iff

$$\sum_{\nu=1}^{\infty} a_\nu = \infty.$$

Solution. From the formula for the n -th convergent

$$\frac{p_n}{q_n} = a_0 + \sum_{\nu=0}^{n-1} (-1)^\nu \frac{1}{q_\nu q_{\nu+1}}$$

it is evident that it suffices to prove:

- a) If $\sum_{\nu=1}^{\infty} a_{\nu} < \infty$ then (q_n) remains bounded for $n \rightarrow \infty$.
- b) If $\sum_{\nu=1}^{\infty} a_{\nu} = \infty$, then $\lim_{n \rightarrow \infty} q_n q_{n+1} = \infty$.

Proof of a) Let $c := \max(q_0, q_1)$. We will show by induction on $n \geq 1$, that

$$q_n \leq c \prod_{\nu=1}^n (1 + a_{\nu}).$$

This is trivial for $n = 1$.

Induction step. From the recursion formula $q_{n+1} = a_{n+1}q_n + q_{n-1}$ and the induction hypothesis one gets

$$q_{n+1} \leq a_{n+1} \prod_{\nu=1}^n (1 + a_{\nu}) + \prod_{\nu=1}^{n-1} (1 + a_{\nu}) \leq \prod_{\nu=1}^{n+1} (1 + a_{\nu}), \quad \text{q.e.d.}$$

Proof of b) Let $c_0 := \min(q_0, q_1)$. From the recursion formula $q_n = a_n q_{n-1} + q_{n-2}$ we get for even $n = 2k$

$$q_{2k} \geq c_0 \quad \text{and} \quad q_{2k} \geq c_0 \sum_{\ell=1}^k a_{2\ell}$$

and for odd $n = 2k + 1$

$$q_{2k+1} \geq c_0 \quad \text{and} \quad q_{2k+1} \geq c_0 \sum_{\ell=1}^k a_{2\ell+1}.$$

Since $\sum_{\nu=1}^{\infty} a_{\nu} = \infty$, at least one of the infinite series $\sum_{\ell=1}^k a_{2\ell}$ or $\sum_{\ell=1}^k a_{2\ell+1}$ diverges.

In the first case, $\lim_{k \rightarrow \infty} q_{2k} = \infty$. Since $q_{2k+1} \geq c_0$ for all k , it follows that $\lim_{n \rightarrow \infty} q_n q_{n+1} = \infty$.

The second case is handled analogously, q.e.d.

Problem 7 Let A be the set of all irrational numbers $0 < x < 1$ whose decimal expansion

$$x = \sum_{\nu=1}^{\infty} c_{\nu} 10^{-\nu}$$

satisfies $c_{\nu} \neq 9$ for all ν . Prove that A has Lebesgue measure 0.

Solution. Let A_n be the set of all real numbers $0 \leq x < 1$ whose decimal expansion

$$x = \sum_{\nu=1}^{\infty} c_{\nu} 10^{-\nu}$$

satisfies $c_{\nu} \neq 9$ for $1 \leq \nu \leq n$.

We will show that the Lebesgue measure of A_n satisfies

$$\lambda(A_n) \leq (9/10)^n.$$

Let X_n be the set of all numbers

$$x = \sum_{k=1}^n c_k 10^{-k} \quad \text{with } c_k \in \{0, 1, 2, \dots, 8\}.$$

The number of elements in X_n is $\#X_n = 9^n$ and we have

$$A_n \subset \bigcup_{x \in X_n} [x, x + 10^{-n}[.$$

Therefore

$$\lambda(A_n) \leq 9^n \cdot 10^{-n} = \left(\frac{9}{10}\right)^n.$$

Since $A \subset \bigcap_{n \geq 1} A_n$, it follows

$$0 \leq \lambda(A) \leq \lim_{n \rightarrow \infty} \lambda(A_n) = 0, \quad \text{q.e.d.}$$

Problem 8 Let M be the set of all positive integers m whose decimal representation

$$m = \sum_{\nu=0}^N c_\nu 10^\nu$$

satisfies $c_\nu \neq 9$ for all ν . Prove that

$$\sum_{m \in M} \frac{1}{m} < \infty.$$

Solution. Let M_n be the set of all n -digit numbers in M , i.e. the set of all positive integers

$$m = \sum_{\nu=0}^{n-1} c_\nu 10^\nu \quad \text{with } c_\nu \in \{0, 1, 2, \dots, 8\}, \quad c_{n-1} \neq 0.$$

We have $\#M_n < 9^n$. The smallest element of M_n is 10^{n-1} . Therefore

$$\begin{aligned} \sum_{m \in M} \frac{1}{m} &= \sum_{n=1}^{\infty} \left(\sum_{m \in M_n} \frac{1}{m} \right) \\ &\leq \sum_{n=1}^{\infty} 9^n 10^{-n+1} = 9 \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = 9 \cdot \frac{1}{1 - 9/10} = 90 < \infty. \end{aligned}$$