# Riemann Surfaces

### Problem sheet #8

### Problem 29

Suppose  $p_1, \ldots, p_n$  are pairwise distinct points of  $\mathbb C$  and let

$$X := \mathbb{C} \setminus \{p_1, \dots, p_n\}.$$

Prove that  $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^n$ .

Hint. Construct an open covering  $\mathfrak{U} = (U_1, U_2)$  of X such that the  $U_{\nu}$  are connected and simply connected and  $U_1 \cap U_2$  has n+1 connected components.

## Problem 30

- a) Show that  $\mathfrak{U}=(\mathbb{P}^1\smallsetminus\{\infty\},\mathbb{P}^1\smallsetminus\{0\})$  is a Leray covering of  $\mathbb{P}^1$  for the sheaf  $\Omega$  of holomorphic 1-forms on  $\mathbb{P}^1$ .
- b) Prove that

$$H^1(\mathbb{P}^1,\Omega) \cong H^1(\mathfrak{U},\Omega) \cong \mathbb{C}$$

and that the cohomology class of  $\frac{dz}{z} \in \Omega(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \Omega)$  is a basis of  $H^1(\mathbb{P}^1, \Omega)$ .

### Problem 31

Let X be the annulus  $X := \{z \in \mathbb{C} : r < |z| < R\}, \ 0 \leqslant r < R \leqslant \infty.$ 

a) Prove that for every  $g \in \mathcal{E}(X)$  there exists an  $f \in \mathcal{E}(X)$  such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

b) Conclude that  $H^1(X, \mathcal{O}) = 0$ .

### Problem 32

Let  $\Lambda := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$  be a lattice where  $\tau \in \mathbb{C}$  with  $T := \operatorname{Im}(\tau) > 0$ . Let  $X := \mathbb{C}/\Lambda$  be the associated torus and  $p : \mathbb{C} \to X$  the canonical projection.

For real numbers  $t_1 < t_2$  with  $t_2 - t_1 \leqslant T$  define

$$Y(t_1, t_2) := \{ z \in \mathbb{C} : t_1 < \text{Im}(z) < t_2 \}$$

and

$$U(t_1, t_2) := p(Y(t_1, t_2)) \subset X.$$

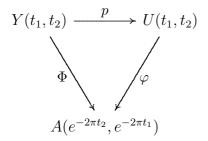
a) Show that

$$\Phi: Y(t_1, t_2) \longrightarrow \mathbb{C}, \quad z \mapsto e^{2\pi i z},$$

maps  $Y(t_1, t_2)$  onto the annulus

$$A(e^{-2\pi t_2}, e^{-2\pi t_1}) := \{ z \in \mathbb{C} : e^{-2\pi t_2} < |z| < e^{-2\pi t_1} \}$$

and that there is a biholomorphic mapping  $\varphi: U(t_1, t_2) \longrightarrow A(e^{-2\pi t_2}, e^{-2\pi t_1})$  which makes the following diagram commutative.



- b) Set  $U_1 := U(0,T)$  and  $U_2 := U(-T/2,T/2)$ . Show that  $\mathfrak{U} := (U_1,U_2)$  is a Leray covering of X for the sheaf  $\mathcal{O}$  and that the intersection  $U_1 \cap U_2$  consists of two connected components  $W_0$  and  $W_1$ .
- c) Prove that

$$H^1(X,\mathcal{O})\cong\mathbb{C}$$

and that the function

$$f_0 \in \mathcal{O}(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \mathcal{O})$$
 with  $f_0 \mid W_0 = 0$  and  $f_0 \mid W_1 = 1$ 

represents a basis of  $H^1(\mathfrak{U}, \mathcal{O}) \cong H^1(X, \mathcal{O})$ .