Multiplicity Structures on Space Curves [1] [2]

C. Bănică and O. Forster

Introduction

Let Y be an analytic (resp. algebraic) curve in a 3-dimensional complex analytic (resp. algebraic) manifold X. In several occasions one has to consider on Y not only the reduced structure, but a "multiplicity structure", which is defined by an ideal $\mathcal{J} \subset \mathcal{O}_X$ with zero set $V(\mathcal{J}) = Y$ but which does not necessarily consist of all functions vanishing on Y. The structure sheaf $\mathcal{O}_X/\mathcal{J}$ of the multiplicity structure may then contain nilpotent elements. For example let Y be a smooth (or more generally locally complete intersection) algebraic curve in affine 3-space $\mathbb{A}^3$. Ferrand/Szpiro (see [6]) have shown that Y is a set-theoretic complete intersection. The two polynomials f, g which describe Y set-theoretically generate an ideal $\mathcal{J}$ which defines a multiplicity 2 structure on Y. For the proof of this theorem, the ideal $\mathcal{J}$ is constructed first in such a way that the conormal module $\mathcal{J}/\mathcal{J}^2$ is globally free of rank 2 and then it follows from a theorem of Serre that $\mathcal{J}$ can be generated by 2 elements.

Another instance where curves with multiplicity structures are useful is in the study of vector bundles of rank 2 on 3-manifolds. Here the curves occur as zero sets of sections of the bundle. These curves carry a natural multiplicity structure. Under some hypotheses one can reconstruct the bundles from the curves (see e. g. [1], [2], [4], [5]).

In this paper, after introducing some notations and conventions, we recall first the Ferrand construction for multiplicity 2 structures and proceed then to a systematic study of structures of higher multiplicity, whose reduction is a smooth curve. Up to multiplicity 4 we obtain a complete description.

§ 0. Notations and generalities

0.1. Although most of the results are also valid in the algebraic case, we work here in the analytic category. By a manifold we mean always a complex-analytic

[1] This is a TEXed version of an article which appeared originally in: Algebraic geometry, Proc. Lefschetz Centen. Conf., Mexico City/Mex. 1984, Part I, Contemp. Math. **58** (1986), 47 – 64.

[2] 1980 Mathematics Subject Classification 32 C, 14 H, 14 B

manifold X. An analytic subspace $Z \subset X$ may be non-reduced, i.e. is a pair $Z = (|Z|, \mathcal{O}_Z)$, where the structure sheaf is of the form $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$, where $\mathcal{I}_Z \subset \mathcal{O}_Z$ is a coherent ideal sheaf with zero-set $|Z|$. For two subspaces Z_1, Z_2 of X write $Z_1 \subset Z_2$ if $\mathcal{I}_{Z_1} \supset \mathcal{I}_{Z_2}$. The intersection $Z_1 \cap Z_2$ is the subspace defined by the ideal $\mathcal{I}_{Z_1 \cap Z_2} := \mathcal{I}_{Z_1} + \mathcal{I}_{Z_2}$.

0.2. In this paper we are mainly concerned with the following situation: There is given a smooth subspace (i.e. submanifold) $Y \subset X$ and another subspace $Z \subset Y$ of X with $|Z| = |Y|$. In a neighborhood of a point $a \in Y$ there exists a holomorphic retraction $X \to Y$, hence also a retraction

$$\pi : Z \to Y,$$

which is the identity on the underlying topological spaces.
(More precisely, one should write $\pi : Z \cap U \to Y \cap U$, U neighborhood of a. But we omit the indication of U for simplicity of notation.)
Now the following conditions are equivalent:

i) Z is Cohen-Macaulay (i.e. all local rings $\mathcal{O}_{Z,z}$ are Cohen-Macaulay)

ii) π is a flat map.

iii) The image sheaf $\pi_*\mathcal{O}_Z$ is locally free over $\mathcal{O}_Y$.

If Y is connected, the rank of $\pi_*\mathcal{O}_Z$ is then constant and equal to the multiplicity of Z.
If Z is Cohen-Macaulay, the multiplicity can be calculated also in the following way: In a neighborhood of a point $a \in Z$ let H be a submanifold of X with $\dim_a Y + \dim_a H = \dim_a X$ and such that H and Y intersect transversally at a. Then the multiplicity of Z at a equals

$$\mu = \dim_{\mathbb{C}} \mathcal{O}_{H \cap Z, a}.$$

0.3. The intersection $H \cap Z$ defines the structure of a multiple point on $\{a\}$. If $\operatorname{codim}_a Y = 2$, H can be considered as a 2-plane. Briançon [3] has classified all multiplicity structures on $0 \in \mathbb{C}^2$ up to multiplicity $\mu = 6$. We give the first cases of his list. For a suitable local coordinate system (x, y) at $0 \in \mathbb{C}^2$, the possible ideals for multiplicity $\leqslant 4$ are

μ	$\mathcal{I}$
1	(x, y)
2	(x, y^2)
3	(x, y^3), (x^2, xy, y^2)
4	(x, y^4), (x^2, y^2), (x^2, xy, y^3)

0.4. A subspace Z of a manifold X is called a locally complete intersectio if for every point $a \in Z$ the ideal $\mathcal{I}_{Z,a}$ can be generated by $r = \operatorname{codim}_a Z$ elements. Locally complete intersections are Cohen-Macaulay.

In the sequel, we will often use the abbreviation CM for Cohen-Macaulay and l.c.i. for locally complete intersection.

§ 1. The Ferrand construction

In this section we recall the Ferrand construction [4] of the doubling of a l.c.i., since this is the basis for our later studies of higher multiplicities.

1.1. Let $Y \subset X$ be a l.c.i. of codimension 2 in a manifold X. The sheaf $\nu_Y := \mathcal{I}_Y/\mathcal{I}_Y^2$ is then locally free of rank 2 over $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$, i.e. corresponds to a vector bundle of rank 2 on Y, which is by definition the conormal bundle of Y. (In the sequel we will identify vector bundles and locally free sheaves.) Now let there be given a line bundle L on Y, i.e. a locally free sheaf of rank 1, and an epimorphism

$$\beta : \nu_Y \to L.$$

Then we can define an ideal $\mathcal{I}_Z \subset \mathcal{O}_X$ with $\mathcal{I}_Y^2 \subset \mathcal{I}_Z \subset \mathcal{I}_Y$ by the following exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Y^2} \longrightarrow \nu_Y \longrightarrow L \longrightarrow 0. \tag{1}$$

An easy calculation shows that $\mathcal{I}_Z$ is again locally generated by two elements: In a neighborhood of a point $y \in Y$ we may choose generators g_1, g_2 of $\mathcal{I}_{Y,y}$ such that their classes $\dot{g}_i := g_i \bmod \mathcal{I}_Y^2 \in \nu_{Y,y}$ satisfy: $\beta(\dot{g}_1) = 0$ and and $\beta(\dot{g}_2)$ is a generator of the stalk L_y. Therefore $(\mathcal{I}_Z/\mathcal{I}_Y^2)_y$ is generated by the class $\dot{g}_1$, hence

$$\mathcal{I}_{Z,y} = (g_1) + \mathcal{I}_{Y,y}^2 = (g_1, g_1^2, g_1 g_2, g_2^2) = (g_1, g_2^2).$$

The subspace $Z = (|Y|, \mathcal{O}_X/\mathcal{I}_Z)$ is called the Ferrand doubling of Y with respect to the epimorphism $\beta : \nu_Y \to L$. (The multiplicity of Z is twice the multiplicity of Y.)

It is clear that two epimorphisms $\beta : \nu_Y \to L$ and $\beta' : \nu_Y \to L'$ define the same subspace Z iff there exists an isomorphism $\varphi : L \to L'$ such that $\beta' = \varphi \circ \beta$.

1.2. Since Z is again a l.c.i., the conormal sheaf $\nu_Z = \mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free, i.e. a vector bundle. We consider its restriction $\nu_Z|_Y := \nu_Z \otimes \mathcal{O}_Y$. We have

$$\nu_Z|_Y = (\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes (\mathcal{O}_X/\mathcal{I}_Y) \cong \mathcal{I}_Z/\mathcal{I}_Y\mathcal{I}_Z.$$

On the other hand, by definition $L = \mathcal{I}_Y/\mathcal{I}_Z$, hence

$$L^2 = (\mathcal{I}_Z/\mathcal{I}_Z^2)^{\otimes 2} \cong \mathcal{I}_Y^2/\mathcal{I}_Y\mathcal{I}_Z.$$

Therefore we get an exact sequence which can be fitted together with (1) to yield the following exact sequence of vector bundles on Y:

$$0 \longrightarrow L^2 \longrightarrow \nu_Z|_Y \longrightarrow \mathcal{I}_Z/\mathcal{I}_Y^2 \longrightarrow 0.$$

From this it follows in particular that

$$\det(\nu_Z|_Y) = \det(\nu_Y) \otimes L. \tag{2}$$

This formula can be used to calculate the dualizing sheaf ω_Z of Z. The dualizing sheaf, which is just the canonical line bundle in the case of a manifold, can be calculated for a l.c.i. Z in a manifold X by the formula

$$\omega_Z = (\omega_X|_Z) \otimes \det(\nu|_Z)^*.$$

Since a similar formula holds for ω_Y, we get from (2)

$$\omega_Z|_Y = \omega_Y \otimes L^{-1}.$$

1.3. If $Y \subset X$ is a submanifold and $Z \supset Y$ a CM-subspace with $|Z| = |Y|$ and multiplicity 2, one can conversely show that $\mathcal{I}_Y^2 \subset \mathcal{I}_Z \subset \mathcal{I}_Y$ and $L := \mathcal{I}_Y/\mathcal{I}_Z$ is locally free of rank 1, hence Z is obtained from Y by the Ferrand construction by means of the natural epimorphism

$$\nu_Y = \mathcal{I}_Y/\mathcal{I}_Y^2 \to \mathcal{I}_Y/\mathcal{I}_Z = L.$$

§ 2. Primitive extensions

2.1. From now on, we consider always the following situation: Let Y be a smooth connected curve in a 3-dimensional manifold X. We are interested in Cohen-Macaulay subspaces Z of X with $Z \supset Y$ and $|Z| = |Y|$.

Such a CM subspace Z is called a *primitive extension* of Y if Z is locally contained in a smooth surface F.

Let us first study the local structure of a primitive extension. We may assume that there is a coordinate system (t, x, y) around the considered point such that F is given by $\mathcal{I}_F = (x)$ and Y is given by $\mathcal{I}_Y = (x, y)$. Since Z is a CM codimension 1 subspace of F, it is given in this coordinate system by $\mathcal{I}_Z = (x, y^{k+1})$ for a certain natural number k. This shows that Z is even a l.c.i. (of multiplicity $k + 1$).

To study the global structure of Z, we define a filtration

$$Y = Z_0 \subset Z_1 \subset \ldots \subset Z_k = Z$$

as follows: We denote by $Y^{(j)}$ the j-th infinitesimal neighborhood of Y in X, given by the ideal $\mathcal{I}_{Y^{(j)}} = \mathcal{I}_Y^{j+1}$ and set

$$Z_j := Z \cap Y^{(j)}, \quad \text{i.e.} \quad \mathcal{I}_{Z_j} = \mathcal{I}_Z + \mathcal{I}_Y^{j+1}.$$

With respect to the local coordinates considered above, we have

$$\mathcal{I}_{Z_j} = (x, y^{j+1}).$$

Thus Z_j is a l.c.i. of multiplicity $j+1$.
%%%TODO Let us assume $k \geqslant 1$. Then we have in particular the extension $Y \subset Z_1$ of multiplicity 2, which can be obtained by the Ferrand construction with the line bundle

$$L = \mathcal{I}_Y / \mathcal{I}_{Z_1} = \mathcal{I}_Y / (\mathcal{I}_Z + \mathcal{I}_Y^2).$$

We will say in this situation that $Z \supset Y$ is a primitive extension of *type* L.

2.2. Proposition. *Let $Z \supset Y$ be a primitive extension of multiplicity $k+1$ and type L. Then one has for $j = 1, ..., k$ exact sequences*

$$0 \longrightarrow L^j \longrightarrow \mathcal{O}_{Z_j} \longrightarrow \mathcal{O}_{Z_{j-1}} \longrightarrow 0,$$

where $Z_j = Z \cap Y^{(j)}$. Further, with the abbreviation $\mathcal{I}_j := \mathcal{I}_{Z_j}$ one has isomorphisms

$$L^j \cong \mathcal{I}_{j+1}/\mathcal{I}_j \cong \mathcal{I}_Y^j / \mathcal{I}_1 \mathcal{I}_Y^{j-1}.$$

Proof. We remark first that $\mathcal{I}_{j-1}/\mathcal{I}_j$ is a locally free $\mathcal{O}_Y$-module of rank 1. This is verified by a local calculation. (In the above coordinates, $\mathcal{I}_{j-1}/\mathcal{I}_j$ is generated by the class of y^j.) On the other band, one has surjective $\mathcal{O}_Y$-morphisms

$$L^j = \left(\frac{\mathcal{I}_Y}{\mathcal{I}_1}\right)^{\otimes j} \xrightarrow{\varphi} \frac{\mathcal{I}_Y^j}{\mathcal{I}_1 \mathcal{I}_Y^{j-1}} \xrightarrow{\psi} \frac{\mathcal{I}_Z + \mathcal{I}_Y^j}{\mathcal{I}_Z + \mathcal{I}_Y^{j+1}} = \frac{\mathcal{I}_{j-1}}{\mathcal{I}_j}$$

Since L^j and $\mathcal{I}_{j-1}/\mathcal{I}_j$ are locally free of rank 1, φ and ψ have to be isomorphisms. □

2.3. Proposition. *Let $Z \supset Y$ be a primitive extension of multiplicity $k+1$ and type L. Then there is an exact sequence*

$$0 \longrightarrow L^{k+1} \xrightarrow{\tau} \nu_Z|_Y \longrightarrow \nu_Y \longrightarrow L \longrightarrow 0.$$

The dualizing sheaf of Z satisfies

$$\omega_Z|_Y = \omega_Y \otimes L^{-k}.$$

Proof. We have

$$\begin{aligned}
&\nu_Z|_Y = (\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes (\mathcal{O}_X/\mathcal{I}_Y) \cong \mathcal{I}_Z/\mathcal{I}_Y\mathcal{I}_Z,\\
&L = \mathcal{I}_Y/\mathcal{I}_1 = \mathcal{I}_Y/(\mathcal{I}_Z + \mathcal{I}_Y^2),\\
&L^{k+1} \cong \mathcal{I}_Y^{k+1}/\mathcal{I}_1\mathcal{I}_Y^k.
\end{aligned}$$

The inclusions

$$\begin{aligned}
&\mathcal{I}_Y^{k+1} \subset \mathcal{I}_Z \subset \mathcal{I}_Y,\\
&\mathcal{I}_1\mathcal{I}_Y^k \subset \mathcal{I}_Y\mathcal{I}_Z \subset \mathcal{I}_Y^2 \subset \mathcal{I}_1
\end{aligned}$$

induce the sequence we are looking for:

$$0 \longrightarrow \frac{\mathcal{I}_Y^{k+1}}{\mathcal{I}_1\mathcal{I}_Y^k} \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Y\mathcal{I}_Z} \longrightarrow \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \longrightarrow \frac{\mathcal{I}_Y}{\mathcal{I}_1} \longrightarrow 0.$$

The exactness is again verified by local calculation. Taking determinants, we get from it

$$\det(\nu_Z|_Y) = \det(\nu_Y) \otimes L^k.$$

This implies

$$\omega_Z|_Y = \omega_Y \otimes L^{-k}.$$

□

Remark. The above formula for ω_Z gives this line bundle only after restriction to Y. Thus one needs information about the restriction map $\mathrm{Pic}(Z) \to \mathrm{Pic}(Y)$. For this we refer to § 3.2.

Now we study the following problem: Let there be given a primitive extension $Z' = Z_{k-1} \supset Y$ of multiplicity $k \geqslant 1$ and type L. Under what conditions can we extend further to a primitive extension $Z \supset Z' \supset Y$ of multiplicity $k+1$? Here we have

2.4. Proposition. *Let $Z' \supset Y$ be a primitive extension of type L and multiplicity k and let*

$$\tau' : L^k \to \nu_{Z'}|_Y$$

be the natural injection (given by Proposition 2.3). Then there is a bijection between the set of primitive extensions $Z \supset Z' \supset Y$ of multiplicity $k+1$ and the set of retractions for τ', i.e. the set of epimorphisms

$$\beta : \nu_{Z'}|_Y \to L^k$$

with $\beta \circ \tau' = \mathrm{id}_{L^k}$. This correspondence is given by the sequence

$$0 \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Y\mathcal{I}_{Z'}} \xrightarrow{\alpha} \frac{\mathcal{I}_{Z'}}{\mathcal{I}_Y\mathcal{I}_{Z'}} = \nu_{Z'}|_Y \xrightarrow{\beta} L^k \longrightarrow 0. \tag{3}$$

Proof. a) Suppose first given a retraction β for τ' and define $\mathcal{I}_Z$ by the exact sequence (3). That $Z \supset Z' \supset Y$ is a primitive extension of multiplicity $k+1$ can be seen locally: In suitable coordinates,

$$\mathcal{I}_Y = (x, y), \quad \mathcal{I}_{Z'} = (x, y^k).$$

In the considered neighborhood, a basis of the bundle $\nu_{Z'}|_Y$ is constituted by the classes $\dot{x}$, $\dot{y}^k$ of x, y^k modulo $\mathcal{I}_Y\mathcal{I}_{Z'}$ and $L^k = \mathcal{I}_Y^k/\mathcal{I}_1\mathcal{I}_Y^{k-1}$ is generated by $e := y^k \operatorname{mod} \mathcal{I}_1\mathcal{I}_Y^{k-1}$. Since β is a retraction, we have

$$\beta(\dot{y}^k) = e, \quad \beta(\dot{x}) = ce.$$

Replacing x by $x' = x - cy^k$, we have $\mathcal{I}_Y = (x', y)$, $\mathcal{I}_{Z'} = (x', y^k)$ and $\beta(\dot{x}') = 0$. Then $\operatorname{Ker}\beta$ is generated by the class of x', hence

$$\mathcal{I}_Z = (x') + \mathcal{I}_Y\mathcal{I}_{Z'} = (x', y^{k+1}),$$

which shows that Z is a primitive extension of multiplicity $k+1$.

b) Conversely, if $Z \supset Z' \supset Y$ is a primitive extension of multiplicity $k+1$, we have $\mathcal{I}_Z \supset \mathcal{I}_Y\mathcal{I}_{Z'}$ and

$$\operatorname{Im}\left(\frac{\mathcal{I}_Z}{\mathcal{I}_Y\mathcal{I}_{Z'}} \xrightarrow{\alpha} \frac{\mathcal{I}_{Z'}}{\mathcal{I}_Y\mathcal{I}_{Z'}}\right)$$

is a subline bundle of $\nu_{Z'}|_Y$, which is the complement of the subline bundle $\operatorname{Im}(\tau') \subset \nu_{Z'}|_Y$ (this is verified by a local calculation). Hence the epimorphism of $\nu_{Z'}|_Y$ to the cokernel of α can be identified with the projection of $\nu_{Z'}|_Y$ onto the summand $\operatorname{Im}(\tau') \cong L^k$ in the direct sum decomposition $\nu_{Z'}|_Y = \operatorname{Im}(\alpha) \oplus \operatorname{Im}(\tau')$.

c) It is clear that different retractions $\beta_1,\ \beta_2 : \nu_{Z'}|_Y \to L^k$ define different ideals $\mathcal{I}_{Z_1},\ \mathcal{I}_{Z_2}$.

□

Remark. For the sequence

$$0 \longrightarrow L^k \xrightarrow{\tau'} \nu_{Z'}|_Y \longrightarrow \nu_Y \longrightarrow L \longrightarrow 0$$

let $M := \operatorname{Ker}(\nu_Y \to L)$. This is a line bundle with $M = \det(\nu_Y) \otimes L^{-1}$. The existence of a retraction for τ' is equivalent to the splitting of the sequence

$$0 \longrightarrow L^k \longrightarrow \nu_{Z'}|_Y \longrightarrow M \longrightarrow 0.$$

Therefore we obtain

2.5. Corollary. *Let $Z' \supset Y$ be a primitive extension of type L and multiplicity $k \geqslant 2$.*

a) *A sufficient condition for the existence of a primitive extension $Z \supset Z' \supset Y$ of multiplicity $k+1$ is*

$$H^1(Y, \det(\nu_Y)^* \otimes L^{k+1}) = 0.$$

b) *If there exists one primitive extension $Z^0 \supset Z' \supset Y$ of multiplicity $k+1$, then the set of all primitive extensions $Z \supset Z' \supset Y$ of multiplicity $k+1$ is in bijective correspondence with*

$$H^0(Y, \det(\nu_Y)^* \otimes L^{k+1}).$$

§ 3. Cohen-Macaulay filtrations, quasi-primitive extensions

3.1. Let Y be a smooth connected curve in a 3-dimensional manifold X and $Z \supset Y$ a CM subspace of X with $|Z| = |Y|$. We will first define the Cohen-Macaulay filtration of the extension $Z \supset Y$. If $Y^{(j)}$ denotes the j-th infinitesimal neighborhood of Y, the intersection $Z \cap Y^{(j)}$ will not be necessarily Cohen-Macaulay, since in the primary decomposition of $\mathcal{I}_{Z \cap Y^{(j)}}$ there might be embedded components. Throwing away all these embedded components, we get a well-defined largest CM subspace

$$Z_j \subset Z \cap Y^{(j)}.$$

Let $k \in \mathbb{N}$ be minimal with $Z \subset Y^{(k)}$, (since Y is connected, k exists). Then of course $Z = Z_k$. The sequence

$$Y = Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_k = Z$$

is called the CM-filtration of Z. One has always $\mathcal{I}_Y^{j+1} \subset \mathcal{I}_{Z_j}$ and there exists a 0-dimensional subset $S \subset Y$ such that

$$\mathcal{I}_{Z_j,y} = \mathcal{I}_{Z,y} + \mathcal{I}_{Y,y}^{j+1} \text{ for all } y \in Y \setminus S \text{ and } j = 0, \ldots, k.$$

For abbreviation let us write $\mathcal{I}_j := \mathcal{I}_{Z_j}$. We assert that

$$\mathcal{I}_Y \mathcal{I}_{j-1} \subset \mathcal{I}_j.$$

This is trivially true in all points $y \in Y \setminus S$, hence $(\mathcal{I}_Y \mathcal{I}_{j-1} + \mathcal{I}_j)/\mathcal{I}_j$ is an ideal in $\mathcal{O}_{Z_j}$ with support contained in S. Since $\mathcal{O}_{Z_j}$ is CM, this ideal must be identically zero, which proves our assertion. Therefore

$$L_j := \mathcal{I}_{j-1}/\mathcal{I}_j$$

are modules over $\mathcal{O}_Y$, which are torsion-free (since $\mathcal{O}_{Z_j}$ is CM), hence locally free. Thus $Z = Z_k$ can be obtained from $Y = Z_0$ by successive extensions

$$0 \longrightarrow L_j \longrightarrow \mathcal{O}_{Z_j} \longrightarrow \mathcal{O}_{Z_{j-1}} \longrightarrow 0, \quad j = 1, \dots, k, \tag{4}$$

by vector bundles L_j. The multiplicity of Z is therefore

$$\mu(Z) = 1 + \sum_{j=1}^{k} \operatorname{rank}(L_j)$$

and we have

$$\chi(Z, \mathcal{O}_Z) = \chi(Y, \mathcal{O}_Y) + \sum_{j=1}^{k} \chi(Y, L_j).$$

3.2. Since $L_j = \mathcal{I}_{j-1}/\mathcal{I}_j$ is an ideal of square zero in $\mathcal{O}_{Z_j}$, we get from (4) exact sequences

$$0 \longrightarrow L_j \longrightarrow \mathcal{O}^*_{Z_j} \longrightarrow \mathcal{O}^*_{Z_{j-1}} \longrightarrow 0,$$

hence exact sequences

$$H^1(Y, L_j) \longrightarrow \operatorname{Pic}(Z_j) \longrightarrow \operatorname{Pic}(Z_{j-1}) \longrightarrow H^2(Y, L_j)$$

from which one can read off sufficient cohomological conditions for the bijectivity of the restriction map $\operatorname{Pic}(Z) \to \operatorname{Pic}(Y)$.

3.3. Analogously to the formula $\mathcal{I}_Y\mathcal{I}_{j-1} \subset \mathcal{I}_j$ one proves $\mathcal{I}_i\mathcal{I}_j \subset \mathcal{I}_{i+j+1}$ for all i, j. This induces a natural multiplicative structure

$$L_i \otimes L_j \to L_{i+j}.$$

In particular, one has morphisms

$$L_1^{\otimes j} \to L_j,$$

which are surjective over $Y \setminus S$.

3.4. We have always a surjective map

$$\nu_Y = \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \longrightarrow \frac{\mathcal{I}_Y}{\mathcal{I}_1} = L_1$$

Hence $\operatorname{rank}(L_1) \leqslant \operatorname{rank}(\nu_Y) = 2$. The case $\operatorname{rank}(L_1) = 0$ is trivial, since this implies $L_j = 0$ for all $j > 0$, hence $Z = Y$. So there remain two non-trivial cases:

i) $\operatorname{rank}(L_1) = 1$,

ii) $\operatorname{rank}(L_1) = 2$.

The second case occurs iff $\mathcal{I}_1 = \mathcal{I}_Y^2$ i.e. $Y^{(1)} \subset Z$. In the first case we will call the extension $Z \supset Y$ *quasi-primitive*. Since generically (i. e. over $Y \setminus S$) we have $\mathcal{I}_1 = \mathcal{I}_Z + \mathcal{I}_Y^2$, the condition $\operatorname{rank}(L_1) = 1$ is equivalent to the condition that generically $\operatorname{emdim}_y Z = 2$. Thus $Z \supset Y$ is a quasi-primitive extension iff it is a primitive extension outside a zero-dimensional subset of Y.

3.5. Let now $Z \supset Y$ be a quasi-primitive extension with CM-filtration

$$Y = Z_0 \subset Z_1 \subset \ldots \subset Z_k = Z$$

and define the bundles $L_j = \mathcal{I}_{j-1}/\mathcal{I}_j$ as above. We will use the abbreviation $L := L_1$. Since the maps $L^j \to L_j$ are generically surjective, it follows that all L_j are line bundles and that there are divisors $D_j \geqslant 0$ on Y such that

$$L_j = L^j(D_j).$$

From the multiplication $L_i \otimes L_j \to L_{i+j}$ we get

$$D_i + D_j \leqslant D_{i+j} \text{ for all } i, j \geqslant 1,$$

where $D_1 := 0$.
Thus to any quasi-primitive extension $Z \supset Y$ we can associate as invariants a line bundle L and a sequence of divisors $D_2, \ldots, D_k$ on Y. We call $(L, D_2, \ldots, D_k)$ the type of the quasi-primitive extension.

3.6. Note that the extension $Z_1 \supset Y$ is obtained by the Ferrand construction using the line bundle L. The other extensions have a more complicated structure. To study them consider the conormal sheaves $\nu_j := \nu_{Z_j} = \mathcal{I}_j/\mathcal{I}_j^2$. We have $\nu_j|_Y = \mathcal{I}_j/\mathcal{I}_Y\mathcal{I}_j$. Since $\mathcal{I}_Y\mathcal{I}_j \subset \mathcal{I}_{j+1}$ and $L_{j+1} = \mathcal{I}_j/\mathcal{I}_{j+1}$ we have an exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_{j+1}}{\mathcal{I}_Y\mathcal{I}_j} \longrightarrow \frac{\mathcal{I}_j}{\mathcal{I}_Y\mathcal{I}_j} = \nu_j|_Y \xrightarrow{\beta_j} L_{j+1} \longrightarrow 0.$$

Thus $\mathcal{I}_{j+1}$ is uniquely determined by I_j and the epimorphism $\beta_j : \nu_j|_Y \to L_{j+1}$. However this epimorphism is not arbitrary, but satisfies a certain condition. To derive this condition, we consider the sequence

$$0 \longrightarrow L^{j+1} \xrightarrow{\tau_j} \nu_j|_Y \longrightarrow \nu_Y \longrightarrow L \longrightarrow 0.$$

As in § 2.3 we have $L^{j+1} = \mathcal{I}_Y^{j+1}/\mathcal{I}_1\mathcal{I}_Y^j$, $\nu_j|_Y = \mathcal{I}_j/\mathcal{I}_Y\mathcal{I}_j$, $\nu_Y = \mathcal{I}_Y/\mathcal{I}_Y^2$, $L = \mathcal{I}_Y/\mathcal{I}_1$ and the maps are induced by the natural inclusions. The sequence is a complex, but not necessarily exact at the places $\nu_j|_Y$ and ν_Y. The composition

$$L^{j+1} \xrightarrow{\tau_j} \nu_j|_Y \xrightarrow{\beta_j} L_{j+1} = L^{j+1}(D_{j+1})$$

is nothing else than the natural inclusion $L^{j+1} \to L^{j+1}(D_{j+1})$.
Thus β_j is a "meromorphic" retraction of τ_j. In a sense, this is the only condition that β_j has to fulfill, as the following proposition shows.

3.7. Proposition. *Let $Z' \supset Y$ be a quasi-primitive extension of type $(L, D_2, \ldots, D_{k-1})$ and multiplicity k and let $\tau' : L^k \to \nu_{Z'}|_Y$ be the natural map induced by the inclusion $\mathcal{I}_Y^k \subset \mathcal{I}_{Z'}$. Let $D_k \geqslant 0$ be another divisor on Y. Then there exists a natural bijective correspondence between the set of quasi-primitive extensions $Z \supset Y$ of multiplicity $k+1$ and type $(L, D_2, \ldots, D_k)$ with CM-filtration $Y = Z_0 \subset Z_1 \subset \ldots \subset Z_{k-1} = Z' \subset Z$ and the set of all epimorphisms*

$$\beta : \nu_{Z'}|_Y \to L^k(D_k)$$

which make commutative the diagram

$$\begin{array}{ccc} \nu_{Z'}|_Y & \xrightarrow{\beta} & L^k(D_k) \\ {\scriptstyle \tau'}\uparrow & \nearrow{\scriptstyle nat} & \\ L^k & & \end{array}$$

Proof. Of course, given β, the associated extension $Z \supset Y$ is defined by the exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Y\mathcal{I}_{Z'}} \longrightarrow \nu_{Z'}|_Y \xrightarrow{\beta} L^k(D_k) \longrightarrow 0$$

By the above remarks it remains only to show that for this Z the maximal CM subspace of $Z \cap Y^{(k-1)}$ coincides with Z'. This is true over $Y \setminus \bigcup \operatorname{Supp}(D_j)$, since there the extension is primitive. Hence it is true everywhere. □

3.8. Parametrization. Assume Y compact. Then, given one β_0 satisfying the conditions of Proposition 3.7, the set of all such β is in bijective correspondence with an open subset of

$$\operatorname{Hom}(K, L^k(D_k)),$$

where $K := (\nu_{Z'}|_Y)/\operatorname{Im}(L^k \to \nu_{Z'}|_Y)$. To determine this set consider the sequence

$$0 \longrightarrow L^k \xrightarrow{\tau'} \nu_{Z'}|_Y \longrightarrow \nu_Y \longrightarrow L \longrightarrow 0.$$

Since this sequence is exact outside a set of dimension zero, $K' := K/\operatorname{Tors}(K)$ is isomorphic to

$$\operatorname{Im}(\nu_{Z'}|_Y \to \nu_Y) \subset M := \operatorname{Ker}(\nu_Z \to L) = \mathcal{I}_1/\mathcal{I}_Y^2.$$

It follows that $K' = M(-D'_{k-1})$, where D'_{k-1} is the divisor determined by

$$\frac{\mathcal{I}_1}{\mathcal{I}_{k-1} + \mathcal{I}_Y^2} \cong \mathcal{O}_{D'_{k-1}}.$$

Since $\operatorname{Hom}(K, L^k(D_k)) = \operatorname{Hom}(K', L^k(D_k))$ and $M = \det(\nu_Y) \otimes L^{-1}$, we see that the set of all β's is parametrized by an open subset of

$$H^0(Y, \det(\nu_Y)^* \otimes L^{k+1}(D'_{k-1} + D_k)),$$

(cf. Corollary 2.5).
Note that $D'_1 = 0$ and that

$$\mathcal{O}_{D_j} = \operatorname{Coker}(L^j \to L_j) \cong \frac{\mathcal{I}_{j-1}}{\mathcal{I}_j + \mathcal{I}_Y^j},$$

hence in particular $D'_2 = D_2$.

3.9. Local structure. Proposition 3.7 can also be used to determine the local structure of quasi-primitive extensions. As an example consider a quasi-primitive extension $Z = Z_2 \supset Z_1 \supset Y$ of multiplicity 3 in the neighborhood of a point $a \in Y$ where $\operatorname{ord}_a(D_2) = d > 0$. Since $Z_1 \supset Y$ is a Ferrand doubling, there exists a local coordinate system (t, x, y) around a such that $\mathcal{I}_Y = (x, y)$, $\mathcal{I}_1 = (x, y^2)$ and $t(a) = 0$. Then $\nu_1|_Y = \mathcal{I}_1/\mathcal{I}_Y\mathcal{I}_1$ is generated by the classes

$$\dot{x} := x \mod \mathcal{I}_Y\mathcal{I}_1, \quad \dot{y}^2 := y^2 \mod \mathcal{I}_Y\mathcal{I}_1,$$

and $L^2 = \mathcal{I}_Y^2/\mathcal{I}_Y\mathcal{I}_1$ is generated by $\dot{y}^2$. In the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathcal{I}_2}{\mathcal{I}_Y\mathcal{I}_1} & \longrightarrow & \dfrac{\mathcal{I}_1}{\mathcal{I}_Y\mathcal{I}_1} & \xrightarrow{\ \beta\ } & L^2(D_2) & \longrightarrow & 0 \\
 & & & & \tau \uparrow & \nearrow \gamma & & & \\
 & & & & L^2 & & & &
\end{array}$$

τ maps $\dot{y}^2$ to $\dot{y}^2$ and γ maps $\dot{y}^2$ to $t^d e$, where e is a local base of $L^2(D_2)$. By the commutativity of the diagram $\beta(\dot{y}^2) = t^d e$. Since β is surjective, we must have $\beta(\dot{x}) = \varphi e$, where $\varphi(0) \neq 0$. Replacing x by $\frac{1}{\varphi}x$, we may suppose $\beta(\dot{x}) = e$. Then $\operatorname{Ker}(\beta)$ is generated by $t^d\dot{x} - \dot{y}^2$, hence

$$\mathcal{I}_{Z_2} = (t^d x - y^2) + \mathcal{I}_Y\mathcal{I}_1 = (t^d x - y^2, xy, x^2).$$

In a similar manner one calculates the local structure of a quasi-primitive extension $Z = Z_3 \supset Z_2 \supset Z_1 \supset Y$ of multiplicity 4 and type (L, D_2, D_3) around a. One gets:

i) If $\text{ord}_a(D_2) = \text{ord}_a(D_3) = d$, then Z_3 is a l.c.i. in a neighborhood of a and

$$\mathcal{I}_{Z_3} = (t^d x - y^2, x^2).$$

If globally $D_2 = D_3 =: D$, then Z_3 is a l.c.i. everywhere and one calculates for the dualizing sheaf

$$\omega_{Z_3}|_Y = \omega_Y \otimes L^{-3}(-D).$$

ii) If $\text{ord}_a(D_2) = d < \text{ord}_a(D_3) = d + \delta$, then Z_3 is not a l.c.i. and in suitable coordinates

$$\mathcal{I}_{Z_3} = (t^\delta(t^d x - y^2) - xy, y(t^d - y^2), x^2).$$

§ 4. Thick extensions of multiplicity 4

4.1. As always, let Y be a smooth connected curve in a 3-dimensional manifold X. A CM-extension $Z \supset Y$ which is not quasi-primitive contains by § 3.4 the full first infinitesimal neighborhood $Y^{(1)}$ of Y. Therefore we will call it a thick extension. In particular, if $Z \supset Y$ is a thick CM-extension of multiplicity 4, we have $Y^{(1)} \subset Z \subset Y^{(2)}$, i.e.

$$\mathcal{I}_Y^3 \subset \mathcal{I}_Z \subset \mathcal{I}_Y^2$$

and $L := \mathcal{I}_Y^2/\mathcal{I}_Z$ is locally free of rank 1. Thus we have an exact sequence

$$0 \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Y^3} \longrightarrow \frac{\mathcal{I}_Y^2}{\mathcal{I}_Y^3} \longrightarrow L \longrightarrow 0.$$

Conversely, let L be a given line bundle on Y and

$$\lambda : \mathcal{I}_Y^2/\mathcal{I}_Y^3 = \mathrm{S}^2\,\nu_Y \longrightarrow L$$

an epimorphism. Then $\text{Ker}(\lambda)$ can be written in the form $\mathcal{I}_Z/\mathcal{I}_Y^3$ and $\mathcal{I}_Z$ defines a CM-extension $Z \supset Y$ of multiplicity 4 with $Y^{(1)} \subset Z$.

4.2. We study now the problem under what conditions on λ the structure Z will be l.c.i. For this purpose we consider more generally a bundle F of rank 2 on Y. One has the squaring map

$$q : F \longrightarrow \mathrm{S}^2\,F.$$

Its image is a quadratic cone $Q \subset \mathrm{S}^2\,F$. If e_1, e_2 is a local base of F and $e_1,\ e_1e_2,\ e_2$ the associated base of the second symmetric powers $\mathrm{S}^2\,F$, then Q consists of all linear combinations $\xi_1 e_1^2 + \xi_2 e_1 e_2 + \xi_3 e_2^2$ such that $4\xi_1\xi_3 - \xi_2^2 = 0$. Let now

$$\lambda : \mathrm{S}^2\,F \longrightarrow L$$

be an epimorphism of $\mathrm{S}^2 F$ onto a line bundle L on Y. We define a discriminant $\mathrm{disc}(\lambda)$ as follows: Let e be a basis of L over some open subset $U \subset Y$ and let e_1, e_2 be a basis of F over U as above. Then λ defines functions $a,\ b,\ c$ on U by

$$\lambda(e_1^2) = ae, \quad \lambda(e_1 e_2) = be, \quad \lambda(e_2^2) = ce.$$

With respect to the given bases, $\mathrm{disc}(\lambda)$ is given by $ac - b^2$. The transformation behavior under base changes of F and L shows then, that $\mathrm{disc}(\lambda)$ is a well defined element

$$\mathrm{disc}(\lambda) \in \Gamma(Y, \det(F)^{-2} \otimes L^2).$$

The discriminant has the following significance: $\mathrm{disc}(\lambda)$ vanishes in a point $p \in Y$ if and only if in the fiber $\mathrm{S}^2 F_p$ the kernel $\mathrm{Ker}(\lambda)_p$ is tangent to the quadratic cone. Now we apply this to the bundle $F = \nu_Y$.

4.3. Proposition. *Let $\lambda : \mathrm{S}^2 \nu_Y \longrightarrow L$ be an epimorphism onto a line bundle L on Y and let $Z \supset Y$ be defined by the exact sequence*

$$0 \longrightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Y^3} \longrightarrow \mathrm{S}^2 \nu_Y \xrightarrow{\lambda} L \longrightarrow 0.$$

Then Z is a l.c.i. at a point $p \in Y$ iff $\mathrm{disc}(\lambda)(p) \neq 0$.

Proof. a) If $\mathrm{disc}(\lambda)(p) \neq 0$, then in the fiber $(\mathrm{S}^2 \nu_Y)_p$ the kernel $\mathrm{Ker}(\lambda)_p$ intersects the quadratic cone in two different lines. Therefore there exist over some neighborhood of p two subline bundles $M_1,\ M_2 \subset \nu_Y$ such that $Q \cap \mathrm{Ker}(\lambda) = q(M_1) \cup q(M_2)$. Choose a basis e_1, e_2 of ν_Y such that e_i is a basis of M_i. Then $\mathrm{Ker}(\lambda)$ is generated by e_1^2 and e_2^2. We can choose local coordinates (t, x, y) in X around p such that $e_1 = x \bmod \mathcal{I}_Y^2$ and $e_2 = y \bmod \mathcal{I}_Y^2$. Then it is easily verified that $\mathcal{I}_Z = (x^2, y^2)$, so Z is a l.c.i. in a neighborhood of p.

b) If $\mathrm{disc}(\lambda)(p) = 0$, we have to distinguish two cases:

i) $\mathrm{disc}(\lambda)$ vanishes identically in a neighborhood of p. This implies $Q \cap \mathrm{Ker}(\lambda) = q(M)$ for some subline bundle M of ν_Y over a neighborhood of p. Then for some basis $e_1 \in M$, e_2 of ν_Y, $\mathrm{Ker}(\lambda)$ is generated by e_1^2, $e_1 e_2$. For a suitable coordinate system (t, x, y) around p we have then

$$\mathcal{I}_Z = (x^2, xy) + (x, y)^3 = (x^2, xy, y^3),$$

which shows that Z is not a l.c.i.

ii) $\mathrm{disc}(\lambda)(p)$ vanishes at p of a certain finite order $d > 0$. If (a, b, c) are the coordinates of λ with respect to some basis $e_1,\ e_2$ of ν_Y and e of L over a neighborhood of p, we have therefore $ac - b^2 = t^d$, where t is a local coordinate on Y with $t(p) = 0$. Since a, b, c cannot simultaneously vanish at p, we have $a(p) \neq 0$ or $c(p) \neq 0$. We may suppose $a(p) \neq 0$. Multiplying e_1 by an invertible function, we may even assume $a \equiv 1$. We replace now e_2 by $e_2' = e_2 - be_1$. Then

$$\lambda(e_1 e_2') = \lambda(e_1 e_2) - b\lambda(e_1^2) = b - b = 0.$$

Hence we may also assume without loss of generality that $b = 0$. Then $c = t^d$ and $\mathrm{Ker}(\lambda)$ is generated by $e_1 e_2,\ e_2^2 - t^d e_1^2$. For a suitable coordinate system (t, x, y) around p we have then

$$\mathcal{I}_Z = (xy, y^2 - t^d x^2) + (x, y)^3 = (xy, y^2 - t^d x^2, x^3),$$

which shows again that Z is not a l.c.i.

□

4.4. Remark. From Proposition 4.3 it follows in particular: If Z is a locally complete intersection everywhere, then the bundle $\det(\nu_Y)^{-2} \otimes L^2$ must be trivial.

As an example let us consider the case $X = \mathbb{P}_3$, $Y = \mathbb{P}_1 \subset \mathbb{P}_3$. Then $\nu_Y = \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y(-1)$. Thus for a thick l.c.i. structure $Z \supset Y$ of multiplicity 4 we have $L = \mathcal{O}_Y(-2)$. The epimorphism

$$\mathrm{S}^2\,\nu_Y = \mathcal{O}_Y(-2)^3 \xrightarrow{\ \lambda\ } \mathcal{O}_Y(-2)$$

is then given by a triple of constants a, b, c with $ac - b^2 \neq 0$ and it is easy to see that there exist (global) homogeneous coordinates (u, v, x, y) on $\mathbb{P}_3$ such that

$$\mathcal{I}_Y = (x, y), \quad \mathcal{I}_Z = (x^2, y^2).$$

Thus Z is a global complete intersection.

4.5. Proposition. *Let $Z \supset Y$ be a thick l.c.i. extension of multiplicity 4 given by an epimorphism $\lambda : \mathrm{S}^2\,\nu_Y \longrightarrow L$. Then we have for the dualizing bundle $\omega_Z|_Y \cong \omega_Y \otimes L^{-1}$.*

Proof. There is an epimorphism

$$\nu_Z|_Y = \mathcal{I}_Z/\mathcal{I}_Y\mathcal{I}_Z \longrightarrow \mathcal{I}_Z/\mathcal{I}_Y^3 = \mathrm{Ker}(\mathrm{S}^2\,\nu_Y \to L),$$

which must be an isomorphism, since both sheaves are locally free $\mathcal{O}_Y$-modules of rank 2. Thus we have an exact sequence

$$0 \longrightarrow \nu_Z|_Y \longrightarrow \mathrm{S}^2\,\nu_Y \longrightarrow L \longrightarrow 0,$$

from which it follows that $\det(\nu_Z|_Y) \cong \det(\mathrm{S}^2\,\nu_Y) \otimes L^{-1} \cong \det(\nu_Y)^3 \otimes L^{-1}$. Since Z is a l.c.i., we have $\det(\nu_Y)^2 = L^2$, hence $\det(\nu_Z|_Y) \cong \det(\nu_Y) \otimes L$, from which the assertion follows. □

Bibliography

[1] C. Bănică and N. Manolache, *Rank* 2 *stable vector bundles on* $\mathbb{P}^3(\mathbb{C})$ *with Chern classes* $c_1 = 1,\ c_2 = 4$, Math. Z. **190** (1985), 315-339.

[2] C. Bănică and M. Putinar, *On complex vector bundles on rational threefolds*, Proc. Cambridge Math. Soc. **97** (1985), 279-288.

[3] J. Briançon, *Description of* $\mathrm{Hilb}^n\,\mathbb{C}\{x, y\}$, Invent. Math. **41** (1977), 45-89.

[4] D. Ferrand, *Courbes gauches et fibrés de rang* 2, C.R. Acad. Sci. Paris **281** (1975), 345-347.

[5] R. Hartshorne, *Stable vector bundles of rank* 2 *on* $\mathbb{P}^3$, Math. Ann. **238** (1978), 229-280.

[6] L. Szpiro, *Equations defining space curves*, Tata Institute Lecture Notes (1979).

C. Bănică
Department of Mathematics
INCREST
bd. Pacii 220
79622 Bucharest
Romania

O. Forster
Mathematisches Institut der LMU
Theresienstraße 39
80333 München,
Germany
Email: forster@math.lmu.de