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# Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)
# Problem sheet 9

1. Let $TM = V \oplus H$.

   a) Show that there exists an affine connetction $\nabla$ such that $\nabla\mathcal{H} = 0$.

   b) Now assume that both $V$ and $H$ are integrable. Show that there is an affine connection which is torsion free and satisfies $\nabla\mathcal{H} = 0$.

2. Show in the following cases that the foliation $\mathcal{F}$ of dimension one is geodesible.

   a) $\mathcal{F}$ is spanned by the Reeb vector field of a contact form $\alpha$ on $M$.

   b) Let $\varphi : M \to M$ be a diffeomorphism. We consider $N = M \times [0,1]/\sim$ where $(x,0) \sim (\varphi(x),1)$. Now let $\mathcal{F}$ be induced by the second factor in $M \times [0,1]$.

3. Let $S^3$ and $S^2$ carry their standard metrics. Consider the Hopf fibration $\pi : S^3 \to S^2$ and the Hopf vectorfield $H$ on $S^3$. ($H$ has length one and is tangent to the fibers of $\pi$.) Let $Z$ be the gradient vector field of $f(x,y,z) = z$ on $S^2$. Define $\mathcal{C} = H^\perp \subset TS^3$ and let $W$ by the unique vector field which is tangent to $\mathcal{C}$ and $D\pi(W) = Z$.

   Let $\mathcal{F}$ be the foliation on $S^3$ spanned by $X = H + W$. To prove that $\mathcal{F}$ is not geodesible we proceed by contradiction. By the theorem of Wadsley, if $\mathcal{F}$ is geodesible, there is a vector field $V$ tangent to $\mathcal{F}$ and a one–form $\alpha$ such that

   $$i_V\alpha = 1 \text{ and } i_V d\alpha = 0 \ . \tag{1}$$

   a) Show that $\mathcal{F}$ has exactly two closed orbits. Describe $\mathcal{F}$ and the asymptotic behaviour of the flow of $X$.

   b) Show that $\alpha \wedge d\alpha = 0$.
   You can proceed as follows: Show that the flow $\varphi$ of $V$ preserves $\alpha \wedge d\alpha$. Fix a volume form $\mu$ on $S^3$ such that

   $$-\mu < \alpha \wedge d\alpha < \mu \ .$$

   Now we argue by contradicton. Assume

   $$\alpha \wedge d\alpha \neq 0$$

   at $p \in S^3$ and $p$ does not lie in a closed orbit of $X$. Let $U$ be a small neighbourhood of $p$. Use $\mu$ to show

   $$\lim_{t\to\infty} \int_{\varphi_t(U)} \alpha \wedge d\alpha = 0$$

   and conlude that $\alpha \wedge d\alpha = 0$ everywhere.

c) Use (1) to show that $d\alpha = 0$. Find a contradiction to the geodesibility of $\mathcal{F}$.

4. Let $\rho : \pi_1(B) \to \mathrm{Diff}(F)$ be a homomorphism. Let $\mathcal{F}$ be the horizontal foliation of the associated flat bundle $M = (\tilde{B} \times F)/\pi_1(B) \to B$. (As usual, $\tilde{B}$ is the universal covering of $B$.)

   a) Prove that the horizontal foliation on $M$ is geodesible and exhibit Riemannian metrics for which it is totally geodesic.

   b) Now assume $\mathrm{Im}(\rho) \subset \mathrm{Isom}(F, g)$ for some Riemmanian metric $g$ on $F$. What can be said about the geodesic and Riemannian properties of the horizontal and vertical foliations on $M$.

Please hand in your solutions during the lecture on Wednesday, 18.12.2002.