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Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)

Problem sheet 6

1. Let C be a contact structure on a manifold M of dimension $2m+1$.

   a) Prove that if m is odd, then M has a preferred orientation and if m is even, C viewed as vector bundle has a preferred orientation. (Hint: Use local defining forms α and define local orientations using $\alpha \wedge (d\alpha)^m$ respectively $d\alpha^m$.)

   b) Show that $\mathbb{RP}^2 \times S^1$ does not admit a contact structure.

   c) Prove that the following two statements are equivalent.

      (i) C is coorientable.

      (ii) C can be defined by a *global* one–form α.

2. Let C be a coorientable contact structure on M. We fix a complement $C^\perp$ of C in TM and a section Z of $C^\perp$ without zeroes. This section defines a trivilization and an orientation of $C^\perp$.

   a) Construct from

   $$\{\alpha \in \Omega^1(M) | \alpha \text{ is a global defining form of } C \text{ and } \alpha(Z) > 0\}$$

   a conformal symplectic structure, i.e. find a non degenerate two–form ω on C which is well defined up to multiplication with positive functions.

   b) Choose a metric g on C. For $X \in C$ consider $\ker(i_X\omega) \subset C$. Use the g–orthogonal complement of this space and the coorientation of C for the construction of JX where J is an almost complex structure on C, i.e. find a bundle endomorphism $J : C \to C$ with $J^2 = -\mathrm{Id}$. Show that this does not depend on the particular choice of ω in its conformal class.

   c) Show that J is independent of the choice of g up to homotopy. How does J depend on the section Z of $C^\perp$ (i.e. the coorientation of C) ?

3. Let D be an oriented Engel structure on an oriented four–manifold M. We denote $D^2 = [D, D]$ by E. Recall that this is an even contact structure.

   a) Show (without using the orientations of D or M) that E is orientable. (Hint : Use local sections X, Y of D and consider the local orientations $X, Y, [X, Y]$.)

b) We denote by W the characteristic line field of E, i.e. if α is a defining form for E we set $W = \ker \left(d\alpha\big|_E \right)$. Show that this implies $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$ where $\mathcal{W}$ respectively $\mathcal{E}$ are the local sections of W respectively E.

c) Prove $W \subset D$ everywhere. (Hint : Argue by contradiction and note $[D, D] = E$. Use that E is bracket–generating.)

d) Choose a complement $W^\perp$ of W in E. Fix a global defining form α for E. Show that $d\alpha\big|_{W^\perp}$ is an orientation of $W^\perp$. Conclude that W is orientable.

e) Show that in the decomposition of TM into the direct sum

$$W \oplus \frac{D}{W} \oplus \frac{E}{D} \oplus \frac{TM}{E}$$

of four line bundles each of these line bundles is orientable and therefore trivial. Conclude that M is parallelizable.

Summary : An orientable 4–manifold which admits an orientable Engel distribution has trivial tangent bundle.

4. Let ∇ be a connection on a distribution $D \subset TM$.

a) Show that the torsion

$$\begin{aligned} T : \mathfrak{X}(D) \times \mathfrak{X}(D) &\longrightarrow \mathfrak{X}(M) \\ (X, Y) &\longmapsto \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

is a tensor.

b) Suppose $S : \mathfrak{X}(D) \times \ldots \times \mathfrak{X}(D) \longrightarrow \mathfrak{X}(D)$ is a tensor of type $(k, 1)$. Show that

$$\begin{aligned} \nabla S : \mathfrak{X}(M) \times \mathfrak{X}(D) \times \ldots \times \mathfrak{X}(D) &\longrightarrow \mathfrak{X}(D) \\ (X_0, X_1, \ldots, X_k) &\longmapsto \nabla_{X_0} \left(S(X_1, \ldots, X_k) \right) \\ &\qquad - \sum_{i=1}^{k} S\left(X_1, \ldots, \nabla_{X_0} X_i, \ldots, X_k\right) \end{aligned}$$

is a tensor.

c) Show that

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(D) &\longrightarrow \mathfrak{X}(D) \\ (X, Y, Z) &\longmapsto -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \end{aligned}$$

is a tensor.

Please hand in your solutions during the lecture on Wednesday, 27.11.2002.