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**Foliations and Subriemannian Geometry, WS 02/03,
(Prof. Kotschick)
Problem sheet 5**

1. Denote the coordinates on $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ by w, x, y, z. Consider the family of 1–forms
$$\alpha_t = tdw + \cos(z)dx - sin(z)dy \ .$$
   a) Show that for all t, the distribution $E_t = \ker(\alpha_t)$ is an even contact structure on T^4.
   b) Prove that the vector field
$$W_t = \frac{\partial}{\partial w} - t\cos(z)\frac{\partial}{\partial x} + t\sin(z)\frac{\partial}{\partial y}$$
   spans the kernel of the restriction of $d\alpha_t$ to E_t.
   c) Describe the foliation defined by W_t on $T^2 = \{y = z = 0\}$.
   d) What is the foliation on T^4 defined by W_0 ?
   e) Show that for every $\varepsilon > 0$ there exists $t_\varepsilon \in (0, \varepsilon)$ such that the even contact structures E_{t_ε} and E_0 are not diffeomorphic, i.e. there is no diffeomorphism ϕ of T^4 such that $\phi_* E_0 = E_{t_\varepsilon}$.

2. Let $P \to B$ be a $U(n)$–principal bundle and $\alpha \in \Omega^1(P)$ a connection form. Recall that the Lie–algebra $\mathfrak{su}(n)$ of $SU(n)$ consists of the trace free matrices in $\mathfrak{u}(n) = \left\{ A \in \mathrm{Mat}(n \times n, \mathbb{C}) \text{ with } \overline{A}^T = -A \right\}$.
   a) Consider the action of S^1 on $\tilde{P} = P/SU(n)$ defined by
$$\begin{aligned} P/SU(n) \times S^1 &\longrightarrow P/SU(n) \\ ([p], \varphi) &\longmapsto \left[p \cdot \mathrm{diag}\left(\exp\left(\frac{i\varphi}{n}\right), \dots, \exp\left(\frac{i\varphi}{n}\right) \right) \right] \ . \end{aligned}$$
   Show that with this action, $\tilde{P} \to B$ is a S^1–principal bundle.
   b) Denote the projection $P \to \tilde{P}$ by pr. Let $\tilde{\alpha}$ be defined as follows. For $\tilde{X} \in T_q\tilde{P}$ choose $X \in TP$ such that $\mathrm{pr}_*(X) = \tilde{X}$. Then let
$$\tilde{\alpha}(\tilde{X}) = \mathrm{tr}(\alpha(X)) \ .$$
   Prove that this is well defined and that $\tilde{\alpha}$ is an S^1–connection on $\tilde{P}$.
   c) Show that the curvature $\tilde{\Omega}$ of $\tilde{\alpha}$ and the curvature Ω of α are related by
$$\mathrm{pr}^* \tilde{\Omega} = \mathrm{tr}(\Omega) \ .$$

3. Consider the embedding $\iota : SO(m) \hookrightarrow SO(m+1)$ defined by

$$A \longmapsto \begin{pmatrix} 1 & 0 \dots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}$$

Let $\mathrm{pr}_1 : SO(m+1) \to S^m$ map an orthogonal matrix to its first column. Furthermore, let $f : \mathfrak{so}(m+1) \to \mathfrak{so}(m)$ be the map which forgets the first row and the first column of matrices in $\mathfrak{so}(m+1)$.

a) Show that with the right action

$$\begin{aligned} SO(m+1) \times SO(m) &\longrightarrow SO(m+1) \\ (B, A) &\longmapsto B \cdot \iota(A) \ , \end{aligned}$$

the bundle $\mathrm{pr}_1 : SO(m+1) \to S^m$ is a principal $SO(m)$–bundle.

b) Let $X \in T_g SO(m+1)$ and $L_{g^{-1}}$ is the left multiplication with g^{-1} on $SO(m+1)$. Then let

$$\alpha(X) = f\left(L_{g^{-1}*} X\right) \ .$$

Show that α is an $SO(m)$–connection on $SO(m+1)$.

c) Let $\underline{X}, \underline{Y}$ be the fundamental vector fields of $X, Y \in \mathfrak{so}(m+1)$ of the right action of $SO(m+1)$ on itself. Show that

$$\Omega(\underline{X}, \underline{Y}) = [f(X), f(Y)] - f([X, Y]) \ .$$

4. We use the construction of flat bundles from exercise 4 on the 2. problem sheet and the notation introduced there. Let G be a Lie group.

a) Show that a principal G–bundle is trivial if and only if it admits a section.

b) Suppose $\rho : \pi_1(B) \longrightarrow G$ is a representation with finite image. Show that there is a finite cover $\overline{B} \to B$ such that the pullback to $\overline{B}$ of the flat principal G–bundle $P \to B$ given by ρ is trivial.

Please hand in your solutions during the lecture on Wednesday, 20.11.2002.