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Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)

Problem sheet 4

1. Let $S^{2m-1} \subset \mathbb{C}^m$ and $S^1 \subset \mathbb{C}$. Define an equivalence relation $\sim$ on S^{2m-1} by $(z_1, \ldots, z_m) \sim (z_1', \ldots, z_m')$ if and only if there exists a complex number $\lambda \in S^1$ such that

$$(\lambda z_1, \ldots, \lambda z_m) = (z_1', \ldots, z_m') \ .$$

The quotient $S^{2m-1}/\sim$ is called B. We consider the Hopf fibration $\pi : S^{2m-1} \to B$.

   a) Show that B is a manifold and identify it. Give an explicit S^1–action on S^{2m-1} such that the Hopf fibration together with this action is a principal S^1–bundle.

   b) Denote $z_j = x_j + iy_j$. Prove that $\alpha = \sum_{j=1}^m (x_j dy_j - y_j dx_j)$ is a S^1–connection on S^{2m-1}.

   c) Calculate the 2–form $\omega \in \Omega^2(B)$ with the property $d\alpha = \pi^*\omega$.

2. On a compact manifold M, let ω_t be a smooth family of k-forms and let X_t be a smooth family of vector fields with $t \in [0,1]$. Recall the formula

$$\left.\frac{d}{dt}\right|_{t=\tau} (\phi_t^*\omega_t) = \phi_\tau^* \left(L_{X_\tau}\omega_\tau + \dot{\omega}_\tau \right) \ . \tag{1}$$

from the last exercise sheet where ϕ_t denotes the flow of X_t. Suppose that there is a smooth family of functions f_t on M such that

$$L_{X_t}\omega_t + \dot{\omega}_t = f_t\omega_t \ .$$

Show that under these assumptions, the family of k–forms defined by

$$\tilde{\omega}_t(p) = \exp\left(\int_{s=0}^t f_s(\phi_s(p))ds \right) \omega_0(p)$$

satisfies the differential equation (1) for $\phi_t^*\omega_t$ with initial value ω_0. Conclude $\phi_t^*\omega_t = \tilde{\omega}_t$.

3. Now we apply the last exercise to contact structures. Let M be a compact manifold of dimension $2m+1$ together with a smooth family of 1–forms α_t such that each α_t defines a contact structure on M. Let $C_t = \ker(\alpha_t)$.

   a) For $X \in C_t(p)$, let $\hat{X}$ be a local section of C_t around $p \in M$ such that $\hat{X}(p) = X$. The subspace of T_p^*M spanned by $\alpha_t(p)$ is denoted by $< \alpha_t(p) >$. Show that the map

   $$\begin{aligned} C_t(p) &\longrightarrow T_p^*M / < \alpha_t(p) > \\ X &\longmapsto \left[L_{\hat{X}} \alpha_t \right] \end{aligned}$$

   is a well defined isomorphism. (Note : The vector space on the right hand side is isomorphic to the dual of C_t.)

   b) Use a) in order to show that there is a unique vector field X_t tangent to C_t and a unique smooth function f_t such that

   $$L_{X_t}\alpha_t + \dot{\alpha}_t = f_t \alpha_t \ .$$

   c) Show that there is a diffeomorphism ϕ_1 of M and a function g_1 such that $\phi_1^* \alpha_1 = g_1 \alpha_0$. (This implies that the contact structures defined by α_1 and α_0 are diffeomorphic.)

4. Let $X_1, \dots, X_k$ be linearly independent vector fields on the $(n-1)$–dimensional sphere S^{n-1}.

   a) Show that the subset

   $$\left\{ (a_1 x_1, \dots, a_q x_q) \,\middle|\, a_i \geq 0 \ , \ \sum_{i=1}^{q} a_i^2 = 1 \text{ and } x_i \in S^{n-1} \right\} \subset \mathbb{R}^{nq}$$

   is S^{nq-1}.

   b) Construct explicitly k linearly independent vector fields on S^{nq-1} from $X_1, \dots, X_k$.

Please hand in your solutions during the lecture on Wednesday, 13.11.2002.