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Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)
Problem sheet 2

1. Let D be the generalised distribution on $\mathbb{R}^2$ spanned by the vector fields $X_1 = \frac{\partial}{\partial x}$ and $X_2 = f(x)\frac{\partial}{\partial y}$, where f is a smooth function vanishing identically for negative x and which is positive for positive x. Determine the accessible sets and growth vectors of D.
2. Let M be a compact manifold with a smooth distribution D. Show that if D is bracket-generating then the length of the growth vectors is uniformly bounded on M.
3. The following construction shows that the growth vector of a distribution can be unbounded even on a compact manifold.

   We denote the length of the growth vector of a given distribution D at p by $l_D(p)$.

   a) Prove that there is a smooth function $f : \mathbb{R}^+ \to \mathbb{R}$ having a zero of order n at n and a zero of order ∞ at $+\infty$.
   b) Let $g(x) = f(1/x)$ if $x > 0$ and $g \equiv 0$ if $x \leq 0$. Show that g is smooth.
   c) For $\alpha = dx + g(y)dz$ consider the distribution $D = \ker \alpha$ on $\mathbb{R}^3$. Show that there is a sequence of points $p_n \in \mathbb{R}^3$ converging to $(0, 0, 0)$ such that the sequence $l_D(p_n)$ is unbounded.
   d) Modifying c) slightly, find a distribution E on T^3 with the property that l_E is an unbounded function on T^3.
4. Let B and F be smooth manifolds and let $\rho\colon \pi_1(B) \to \mathrm{Diff}(F)$ be a representation. We denote the universal covering of B by $\tilde{B}$. Consider the covering $c : \tilde{B} \times F \to (\tilde{B} \times F)/\pi_1(B) = M$. The manifold M is a fibration with fiber F over B. It also carries a foliation $\mathcal{F}$ transverse to the fibers induced by the projection $p : \tilde{B} \times F \to F$. For $f \in F$ the leaf $c(p^{-1}(f))$ is diffeomorphic to $\tilde{B}/\Gamma$ where $\Gamma \subset \pi_1(B)$ is a subgroup.

   a) Characterize the group Γ.
   b) Find conditions on ρ implying that all leaves of $\mathcal{F}$ are diffeomorphic.
   c) Find conditions on ρ implying that $\mathcal{F}$ is induced by a fibration.

The **exercise classes** will take place on Thursdays, 9–11 in room E46.

Please hand in your solutions during the lecture on Wednesday, 30.10.2002.