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Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)

Problem sheet 13

1. Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. For $s \in [0,\infty)$ let $\alpha_s = dx + s dy$ define the foliations $\mathcal{F}_s = \ker(\alpha_s)$. Calculate the cohomologies $H^*(\mathcal{I}_s)$ and $H^*(\mathcal{F}_s)$ for all $s \in [0,\infty)$ as far as You can.

2. Let $\mathcal{F}$ be a foliation of codimension one which can be defined by a global 1–form α on a manifold M. Let $\beta \in \Omega^1(M)$ be such that $d\alpha = \alpha \wedge \beta$. Show that the *Godbillon–Vey class* of $\mathcal{F}$
$$\mathrm{GV}(\mathcal{F}) = \beta \wedge d\beta \in H^3_{\mathrm{dR}}(M)$$
is independent of the choice of α respectively β.

3. Let $G = \mathrm{Sl}(2,\mathbb{R})$ be the Lie–group of real 2×2–matrices with determinant one. The Lie–algebra $\mathfrak{sl}(2,\mathbb{R})$ is generated by
$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
We view A, B, C as left–invariant vector fields on G. Let α, β, γ be the dual frame. Show that $d\alpha = \alpha \wedge (2\beta)$. Hence α defines a G–invariant foliation. Express $\beta \wedge d\beta$ in terms of α, β, γ.

   Remark : On compact quotients of G one obtains in this way a foliation with non–vanishing Godbillon–Vey class.

4. Let M_1, M_2 be two oriented connected manifolds of dimension three and for $i = 1, 2$ let $\mathcal{F}_i$ be a cooriantable foliation of rank two on M_i. We say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *cobordant* if there is a compact oriented 4–manifold X and a foliation $\mathcal{F}$ of codimension 1 on X with the following properties:

   (i) $\partial X = M_1 \cup \overline{M_2}$ where $\overline{M_2}$ is M_2 with the reversed orientation,

   (ii) $\mathcal{F}$ is transversal to ∂X,

   (iii) $TM_i \cap \mathcal{F} = \mathcal{F}_i$ for $i = 1, 2$.

   Show that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are cobordant then
$$\int_{M_1} \mathrm{GV}(\mathcal{F}_1) = \int_{M_2} \mathrm{GV}(\mathcal{F}_2) \ .$$

Please hand in your solutions during the lecture on Wednesday, 30.1.2003.