Mathematisches Institut
Universität München
Thomas Vogel, Zi. 308
thomas.vogel@mathematik.uni-muenchen.de

Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)

Problem sheet 12

1. Let $\mathrm{pr} : M \to B$ be a fibration. Let $\mathcal{F}$ be the foliation whose leaves are the fibers of pr. Find a metric g on M such that $\mathcal{F}$ is Riemannian and minimal with respect to the same metric g.

2. a) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Show that
$$\begin{aligned} \Phi : T^2 &\longrightarrow T^2 \\ (x,y) &\longmapsto (x, x+y) \end{aligned}$$
preserves no metric on T^2. (Hint : Consider the action of Φ on the fundamental group of T^2.)

   b) Consider the representation $\rho : \pi_1(S^1) \simeq \mathbb{Z} \to \mathrm{Diff}(T^2)$ defined by $\rho(1) = \Phi$. Let M_ρ be the total space of the flat bundle $\mathrm{pr} : M_\rho \to S^1$ associated to ρ. Show that the foliation whose leaves are the fibers pr is not geodesible.

3. The following construction is due to W. Thurston. Let (F, σ) and (B, ω) be compact connected symplectic manifolds, $i_b : F \to \pi^{-1}(b)$ for $b \in B$. Assume that $\pi : M \to B$ is a fibre bundle with fiber F for which there is a cohomology class $a \in H^2(M; \mathbb{R})$ with the property $i_b^* a = [\sigma]$. We fix a covering of B by contractible sets U_α, a partition of unitity ρ_α subordinate to this covering and fiber preserving diffeomorphisms $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. Let $\mathrm{pr}_{\alpha 2} : U_\alpha \times F \to F$ be the projection on the second factor and $\sigma_\alpha = \mathrm{pr}_{\alpha 2}^* \sigma$.

   a) Let $\tau_0 \in \Omega^2(M)$ be a representative of a. Show that there are 1–forms λ_α such that $d\lambda_\alpha = \Phi_\alpha^* \sigma_\alpha - \tau_0$. Show that
$$\tau = \tau_0 + \sum_\alpha d\left((\rho_\alpha \circ \pi)\lambda_\alpha\right)$$
is closed and non–degenerate on the fibers of π.

   b) Show that
$$\omega_k = \tau + k\pi^*\omega$$
is a symplectic form on M if k is large enough.

   c) In the situation above let F be an orientable surface such that $0 \neq [F] \in H_2(M; \mathbb{R})$. Find a cohomology class a as in the assumptions of this exercise. Show that the condition $[F] \neq 0$ is also necessary for the existence of a symplectic form on M which restricts to a symplectic form on the fibers.

4. On $\mathbb{R}^4$ consider $\omega = dx \wedge dy + dw \wedge dz$. Let V be the distribution spanned by $\partial_x, \partial_w + e^x \partial_y$. Check that V is symplectic and not integrable. Show that $\overline{\omega}$ (this form was constructed in the lecture) is not V–closed.

Please hand in your solutions during the lecture on Wednesday, 22.1.2003.