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**Foliations and Subriemannian Geometry, WS 02/03,
(Prof. Kotschick)**

Problem sheet 11

1. Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold of dimension $n+k$ and let M be a n–dimensional submanifold. We denote by $\widetilde{\nabla}$ the Levi–Civita connection of $\widetilde{M}$ and $\pi : T\widetilde{M}|_M \to TM^\perp$ is the orthogonal projection along TM. The second fundamental form of M is

$$\begin{aligned} \alpha : TM \times TM &\longrightarrow TM^\perp \\ (U,V) &\longmapsto \pi(\tilde{\nabla}_U V) \ . \end{aligned}$$

   a) Show that the second fundmental form is tensorial, symmetric and bilinear.

   b) Show that the Levi–Civita connection ∇ on M with the metric $g = \widetilde{g}|_M$ is given by

$$\nabla_U V = \widetilde{\nabla}_U V - \alpha(U,V)$$

   for local vector fields U, V on M.

2. We continue considering the situation from exercise 1. Let Y be a vector field on $\widetilde{M}$ which vanishes on ∂M and which is orthogonal to M. The flow of Y is denoted by Φ_t.

   We will denote $\Phi_t\big|_M$ again by Φ_t. Let $g_t := \Phi_t^* g$ be the pull–back metric on M and $dvol_t := dvol_{g_t}$ the associated Riemannian volume form.

   Recall that in local coordinates $x_1, \dots, x_k$ on a Riemanian manifold (N, h), the Riemannian volume form has the following expression :

$$dvol_h = \left[\det\left(h\left(\partial_{x_i}, \partial_{x_j}\right)\right)\right]^{1/2} dx_1 \wedge \ldots \wedge dx_k \ .$$

   a) Find the function $J(m,t)$ such that $dvol_t = J(m,t) dvol_0$.

   b) In order to calculate the first variation

$$\left.\frac{d}{dt}\right|_{t=0} (\mathrm{vol}\Phi_t(M)) = \int_M \frac{\partial}{\partial t} J(m,t) dvol_0$$

   fix a local coordinate system around $m_0 \in M$ such that $x_{n+1} = \ldots x_{n+k} = 0$ locallay defines M. Furthermore, assume that at m_0 the vectors $\partial_1, \dots, \partial_{n+k}$ are orthonormal and oriented. Use these coordinates to expess $\frac{\partial}{\partial t} J(m_0, t)$ and simplify. Eliminate all $\widetilde{\nabla}$.

c) From b) You should get

$$\left.\frac{\partial}{\partial t}\right|_{t=0} J(m_0, t) = -\sum_{i=1}^{n} \widetilde{g}(Y, \alpha(\partial_i, \partial_i))$$

Conclude that if M is minimal with respect to $\widetilde{g}$ then $\sum_{i=1}^{n} \alpha(\partial_i, \partial_i) = 0$. (Note that this condition is independent of the choice of an orthonormal basis of $T_{m_0} M$.)

3. If a submanifold M of $\widetilde{M}$ satisfies only the condition on the second fundamental form from exercise 2, we say that M is critical.

   a) Let $\mathcal{F}$ be a foliation on M which is totally geodesic with respect to g. Show that all leaves of $\mathcal{F}$ are critical.

   b) Show that the equator S^{n-1} in S^n is critical with respect to the round metric. Find a global variation of S^{n-1} such that the volume decreases.

4. Let ω be a closed 1–form without zeroes on M. Let $\mathcal{F}$ be the foliation $\ker(\omega)$. Show that M admits a metric g such that the leaves of $\mathcal{F}$ are minimal and α is harmonic.

   Hint : Use Tischler's theorem to define fibration $\pi : M \longrightarrow S^1$ and use the Moser Lemma (Sheet 3) to find a closed form which is a volume form on the leaves of M.

Please hand in your solutions during the lecture on Wednesday, 15.1.2003.