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Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)

Problem sheet 10

Throughout this exercise sheet, $\mathcal{F}$ will be a smooth foliation of codimension q on the manifold M. The normal bundle of $\mathcal{F}$ is assumed to be oriented.

1. We define the set of basic r–forms $\Omega^r_b(\mathcal{F})$ by

$$\Omega^r_b(\mathcal{F}) := \left\{ \alpha \in \Omega^r(M) \middle| i_X\alpha = 0 \text{ and } i_X d\alpha = 0 \text{ for all } X \in T\mathcal{F} \right\} \ .$$

   The basic cohomology H^*_b is the cohomology of the complex $(\Omega^*_b(\mathcal{F}), d)$.

   a) Show that $\Omega^*_b(\mathcal{F})$ is closed under $\wedge$ and d. In particular, the basic cohomology is well defined.
   b) What can You say about $H^0_b(\mathcal{F})$?
   c) Show that $i : \Omega^1_b(\mathcal{F}) \to \Omega^1(M)$ induces an inclusion $i : H^1_b(\mathcal{F}) \to H^1(M)$.
   d) Suppose $\mathcal{F}$ is given by the fibers of a fibration $M \to B$. What is the basic cohomology $H^*_b(\mathcal{F})$ in this case?

2. The following example is due to E. Ghys. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two–dimensional torus. We suspend the diffeomorphism

$$\begin{aligned} \phi : T^2 &\longrightarrow T^2 \\ (x, y) &\longmapsto (x, x + y) \ , \end{aligned}$$

   i.e. we consider the manifold $T^2 \times [0,1]/\sim$ with $(x, 0) \sim (\phi(x), 1)$. Consider the foliation $\mathcal{G}$ induced by the second factor in $T^2 \times [0, 1]$.

   a) Let ω be a ϕ–invariant 1–form on T^2. Show that ω has the form $f(x)dx$.
   b) Show that ϕ–invariant 2–forms on T^2 have the form $g(x)dx \wedge dy$.
   c) Deduce that $H^2_b(\mathcal{G})$ has infinite dimension. Note that 1c) shows that H^1_b has finite dimension for compact M.

3. Fix a Riemannian metric g on M. At each point of M, choose an oriented orthonormal basis $e_1(p), \ldots, e_q(p)$ for $\mathcal{F}^\perp_p$. Let $e^*_i(p)(\cdot) = g(e_i, \cdot)$. We define the transversal volume form μ^{tr} by

$$\mu^{tr}(p) = e^*_1(p) \wedge \ldots \wedge e^*_q(p) \ .$$

   a) Show that this is a smooth form which depends only on g.

b) Show that if g is bundle–like, then μ^{tr} is a basic q–form. Conclude that it is closed.

4. We define

$$\Omega_0^r(\mathcal{F}) := \left\{\alpha \in \Omega^r(M) \middle| i_X\alpha = 0 \text{ for all } X \text{ tangent to } \mathcal{F}\right\} \ .$$

Note that $\Omega_b^r(\mathcal{F}) \subset \Omega_0^r(\mathcal{F})$. We fix a Riemannian metric g on M, the transversal volume form μ^{tr} was defined in exercise 3.

a) Define a transversal Hodge–$*$–operator, i.e. find a map

$$*_t : \Omega_0^r(\mathcal{F}) \longrightarrow \Omega_0^{q-r}(\mathcal{F})$$

which satisfies $\alpha \wedge *_t\beta = g(\alpha, \beta)\mu^{tr}$ for all $\alpha, \beta \in \Omega_0^r(\mathcal{F})$.

b) Show that if g is bundle–like, $*_t$ maps basic forms to basic forms.

5. Assume that $\mathcal{F}$ is transversly parallelizable, i.e. there exist q vector fields $P_1, \dots, P_q$ which are linearly independant everywhere, no P_i is tangent to $\mathcal{F}$ somewhere and all P_i are basic. Show that $\mathcal{F}$ is Riemannian.

Please hand in your solutions during the lecture on Wednesday, 8.1.2003.