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Foliations and Subriemannian Geometry, WS 02/03, (Prof. Kotschick)
Problem sheet 1

1. Let M be a *paracompact* manifold with a smooth foliation $\mathcal{F}$. Prove that every leaf of $\mathcal{F}$ intersects each foliated chart in at most countably many plaques. (Recall that a topological space is *paracompact* if every open covering has a locally finite refinement.)

2. The following example is due to M. Kneser.

   Let $P = \bigcup_{\alpha\in\mathbb{R}}(\mathbb{R}^3)_\alpha$ be the disjoint union of copies of $\mathbb{R}^3$ and define an equivalence relation on P by
   $$(x_\alpha, y_\alpha, z_\alpha) \sim (x_\beta, y_\beta, z_\beta) \quad \text{if and only if} \quad \begin{cases} x_\alpha = x_\beta \ (= x) \ \neq 0 \\ y_\alpha = y_\beta + \frac{\beta-\alpha}{x} \\ z_\alpha = z_\beta + \frac{(\beta-\alpha)x}{|x|} \end{cases}$$
   for $(x_\alpha, y_\alpha, z_\alpha) \in (\mathbb{R}^3)_\alpha$ and $(x_\beta, y_\beta, z_\beta) \in (\mathbb{R}^3)_\beta$. Set $M = P/\sim$ with the quotient topology.

   a) Prove that the inclusions
   $$\phi_\alpha : \mathbb{R}^3 = (\mathbb{R}^3)_\alpha \subset P \to M$$
   define an atlas for M and find the transition function $\phi_\beta^{-1} \circ \phi_\alpha$.

   b) Show that M is Hausdorff.

   c) Show that the subsets $\{z = \text{const.}\} \subset \mathbb{R}^3$ define a foliation $\mathcal{F}$ on M.

   d) Show that $\mathcal{F}$ has only one leaf. (This shows that M is not paracompact.)

   e) Why do you need the y–coordinate ?

3. Consider the distribution $\mathcal{D} = \ker(dx + ydz)$ on $\mathbb{R}^3$. Show directly that any two points of $\mathbb{R}^3$ can be joined by a smooth unbroken path tangent to $\mathcal{D}$.

4. Calculate the growth vectors of the following distributions at every point

   a) $\mathcal{D} = \ker\ \alpha$ with $\alpha = dx + y^k dz, k \geq 0$ on $\mathbb{R}^3$.

   b) $\mathcal{D} = \ker\ \alpha_1 \cap \ker\ \alpha_2$ on $\mathbb{R}^2 \times T^2$ with
   $$\alpha_1 = dx + \cos\varphi d\theta$$
   $$\alpha_2 = dy + \sin\varphi d\theta\ .$$

5. Consider the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$. Let $\mathcal{D}$ be the generalized distribution $\mathcal{D} = \text{span}\{X, Y\}$. Prove that $Acc(p, \mathcal{D}) = \mathbb{R}^2$ for every $p \in \mathbb{R}^3$.

Please hand in your solutions during the lecture on Wednesday, 23.10.2002.