# A construction of Engel structures

# Une construction des structures d'Engel

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**Abstract**

Every 4–manifold with trivial tangent bundle admits an Engel structure. *To cite this article: Thomas Vogel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

**Résumé**

Toute variété de dimension 4 dont le fibré tangent est trivial admet une structure d'Engel. *Pour citer cet article : Thomas Vogel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

## Version française abrégée

Une structure d'Engel est un champs de plans différentiable $\mathcal{D}$ sur une variété de dimension 4 qui satisfait les conditions

$$\mathrm{rang}[\mathcal{D},\mathcal{D}] = 3 \qquad\qquad \mathrm{rang}[\mathcal{D},[\mathcal{D},\mathcal{D}]] = 4\ .$$

On note ici $[\mathcal{D},\mathcal{D}]$ l'ensemble des vecteurs tangents à $M$ qu'on peut obtenir comme commutateurs $[X,Y]$ pour des sections locales $X,Y$ de $\mathcal{D}$. Les structures d'Engel sont stables au sens de la théorie des singularités. Il y a peu des types des distributions qui soient stables en ce sens. Ce sont les champs des droites, les structures de contact sur les variétés de dimension impaire, les structures de contact paires sur les variétés de dimension paire et les structures d'Engel en dimension 4 [6]. En particulier, les structures d'Engel sont une spécificité de la dimension 4. Ces faits sont une motivation pour l'étude des structures d'Engel.

A chaque structure d'Engel $\mathcal{D}$ sur une variété $M$, on peut associer des distributions

$$\mathcal{W}\subset\mathcal{D}\subset\mathcal{E}\subset TM \tag{1}$$

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où $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ est une structure de contact paire et $\mathcal{W}$ est un champ de droites associé à $\mathcal{E}$. Le feuilletage induit est tangent à $\mathcal{D}$ et est appelé *feuilletage caractéristique* de $\mathcal{D}$. Il ne dépend que de $\mathcal{E}$. Sur une hypersurface $H$ qui est transverse à $\mathcal{W}$ on a une structure de contact $TH \cap \mathcal{E}$ et un champs de droites $TH \cap \mathcal{D}$ tangent à la structure de contact sur $H$. Ces droites sont appelées *droites d'intersection.* Si $M$ est orientée on obtient une orientation de $\mathcal{W}$ et de $TH \cap \mathcal{E}$. Dans ce cas, une orientation de $\mathcal{D}$ induit une orientation des droites d'intersection sur $H$. En utilisant (1) on obtient la proposition suivante :

**Proposition 0.1** *Si $M$ est une variété orientable qui admet une structure d'Engel orientable, le fibré tangent de $M$ est trivial.*

Dans [5] cette proposition est attribuée à V. Gershkovich. Le résultat principal de [8] annoncé dans cette note est la réciproque de la Proposition 0.1.

**Théorème 0.2** *Une variété de dimension quatre dont le fibré tangent est trivial admet une structure d'Engel orientable.*

La preuve du Théorème 0.2 utilise la décomposition d'une variété en anses rondes. Une anse ronde $R_k$ de dimension $n$ et d'indice $k \in \{0, \ldots, n-1\}$ est definie par $R_k = D^k \times D^{n-k-1} \times S^1$. On attache $R_k$ à une variété $M$ à bord en utilisant un plongement de $\partial_- R_k = \partial D^k \times D^{n-k-1} \times S^1$ dans $\partial M$.

Soit $M$ une variété fermée de dimension 4 dont le fibré tangent est trivial. On fixe une décomposition de $M$ en anses rondes. Une telle décomposition existe grâce au théorème suivant.

**Théorème 0.3 (Asimov [1])** *Si $M$ est une variété fermée et connexe de dimension $n \neq 3$, elle admet une décomposition en anses rondes si et seulement si sa caractéristique d'Euler est nulle.*

Sur les anses rondes on fixe des structures d'Engel orientables qui serviront comme modèles. On choisit les modèles tels que leurs feuilletages caractéristiques soient transverses à $\partial_- R_k$ et à $\partial_+ R_k = \partial R_k \setminus \partial_- R_k$. Les feuilletages caractéstiques sont orientées de façon qu'ils sortent de $R_k$ par $\partial_+ R_k$ et rentrent par $\partial_- R_k$.

Soit $M'$ une variété munie d'une structure d'Engel dont le feuilletage caractéristique est transverse au bord et sort le long de $\partial M'$. On fixe une des structures d'Engel modèles sur $R_k$ et un plongement $\psi : \partial_- R_k \to \partial M'$. Si $\psi$ préserve les structures de contact avec leurs orientations et les droites d'intersection orientées on obtient une structure d'Engel différentiable sur $M' \cup_\psi R_k$.

Pour la preuve du Théorème 0.2 on attache les anses rondes dans la décomposition de $M$ en anses rondes l'une après l'autre. On montre qu'à chaque étape on peut modifier les fonctions de recollement des anses rondes et qu'il est possible de choisir une structure d'Engel sur l'anse ronde telle que les conditions sur la fonction de recollement soient satisfaites. Une difficulté importante est de construire un ensemble assez grand de structures d'Engel modèles sur les anses rondes.

Une démonstration detaillée du Théorème 0.2 sera publiée ultérieurement [9].

## 1. Introduction

An Engel structure is a smooth maximally non–integrable plane field on a 4–manifold. Engel structures arise for example as generic germs of plane fields on $\mathbb{R}^4$. The perturbation of an Engel structure is again an Engel structure if the perturbation is small enough in the strong $C^2$–topology. Around every point of a manifold with an Engel structure $\mathcal{D}$, there are coordinates $x, y, z, w$ such that $\mathcal{D}$ is the intersection of the kernels of $dz - x\,dy$ and $dx - w\,dy$. This normal form for Engel structures is due to F. Engel [3]. In particular, Engel structures are stable in the sense of singularity theory.

There are only a few types of distributions with this stability property. In dimension $n$ the stable germs of distributions arise for distributions of rank $k = 1$ or $k = n - 1$ for arbitrary $n$ or if $k = 2$ and $n = 4$, cf. [6]. The case $k = 1$ corresponds to foliations of rank 1 while the case $k = n - 1$ is realized by contact structures if $n$ is odd and by even contact structures if $n$ is even. If $k = 2$ and $n = 4$, the stable distribution germ is an Engel structure. Even among the distributions with stable germs, Engel structures appear to be special due to their exceptional appearance in dimension 4. This motivates the study of Engel structures.

As we explain in Section 2, an Engel structure $\mathcal{D}$ on a 4–manifold $M$ induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM \tag{2}$$

where each distribution has corank 1 in the distribution to its right. Taking orientations into account one can show the following proposition.

**Proposition 1.1** *An orientable 4–manifold which admits an orientable Engel structure has trivial tangent bundle.*

This proposition can be found in [5], where it is attributed to V. Gershkovich. Up to now one can find only few examples of closed Engel manifolds in the literature (cf. [4,6]). The main result of [8] announced in this note is the converse of Proposition 1.1.

**Theorem 1.2** *Every 4–manifold $M$ with trivial tangent bundle admits an orientable Engel structure.*

In the following sections we discuss Theorem 1.2 and another result from [8]. Detailed proofs will be published elsewhere [9].

## 2. Properties of Engel structures

In this section we give the definition of Engel structures and we explain some properties which will be used in the discussion of our results. When $\mathcal{D}, \mathcal{D}'$ are distributions on $M$, then $[\mathcal{D}, \mathcal{D}']$ at $p \in M$ consists of those tangent vectors of $M$ which can be obtained by evaluation of the commutator $[X, X']$ of local sections $X$ of $\mathcal{D}$ and $X'$ of $\mathcal{D}'$ at $p$.

**Definition** 2.1 *A plane field $\mathcal{D}$ on a 4–manifold $M$ is an* Engel structure *if*

$$\operatorname{rank}[\mathcal{D}, \mathcal{D}] = 3 \qquad \textit{and} \qquad \operatorname{rank}[\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = 4 \ .$$

In our discussion we shall also encounter contact structures and even contact structures. Let us recall the definitions.

**Definition** 2.2 *A distribution $\mathcal{C}$ of hyperplanes on a manifold of odd dimension $2k + 1$ is a* contact structure *if it is locally defined by a 1–form $\alpha$ with the property $\alpha \wedge d\alpha^k \neq 0$.*

*A field $\mathcal{E}$ of hyperplanes on a manifold of even dimension $2k$ is an* even contact structure *if it is locally defined by a 1–form $\alpha$ such that the restriction of $d\alpha$ to $\mathcal{E} = \ker(\alpha)$ has maximal rank.*

To each Engel structure $\mathcal{D}$ on $M$ we associate $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$, which is an even contact structure, and the characteristic foliation $\mathcal{W}$ of $\mathcal{E}$. By definition, $\mathcal{W}$ is the only line field tangent to $\mathcal{E}$ which satisfies $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$, i.e. every flow tangent to $\mathcal{W}$ preserves $\mathcal{E}$. Using this property one can show that $\mathcal{W} \subset \mathcal{D}$. Hence we obtain (2).

A classical construction of Engel structures is called prolongation. It is described in the following example.

*Example 1 Let $N$ be a 3–dimensional manifold with contact structure $\mathcal{C}$. Consider the projectivization $\mathbb{P}\mathcal{C}$ of $\mathcal{C}$ consisting of 1–dimensional subspaces of $\mathcal{C}$. Thus $\mathbb{P}\mathcal{C}$ is a fiber bundle over $N$ with fiber $\mathbb{RP}^1$. We denote the bundle projection by $\pi$ and Legendrian lines by $\lambda$. The distribution on $\mathbb{P}\mathcal{C}$*

$$\mathcal{D}_{\mathcal{C}} = \left\{ v \in T_\lambda \mathbb{P}\mathcal{C} \,\middle|\, \pi_*(v) \in \lambda \textit{ for } \lambda \in \mathbb{P}\mathcal{C} \right\}$$

*is an Engel structure. The induced even contact structure is $\pi_*^{-1}(\mathcal{C})$ and the characteristic foliation corresponds to the fibers of $\mathbb{P}\mathcal{C}$.*

Another construction due to H. J. Geiges [4] yields an Engel structure on the mapping torus of a diffeomorphism of a 3–manifold if the mapping torus has trivial tangent bundle.

Let $\mathcal{D}$ be an Engel structure on $M$ and $H$ a hypersurface transverse to the characteristic foliation $\mathcal{W}$. Then $TH \cap \mathcal{E}$ is a distribution of rank 2 and it is easy to show that $TH \cap \mathcal{E}$ is a contact structure. It follows from $\mathcal{W} \subset \mathcal{D}$ and transversality that $TH \cap \mathcal{D}$ is a line field tangent to the contact structure. We refer to $TH \cap \mathcal{D}$ as the *intersection line field* on $H$.

An orientation of $\mathcal{W}$ and $\mathcal{D}$ induces an orientation of the intersection line field. Contact structures in dimension 3 induce an orientation of the underlying manifold. Therefore an orientation of $\mathcal{W}$ together with the contact orientation on local transversals induces an orientation of $M$. If $X, Y$ is a local frame for $\mathcal{D}$, then $X, Y, [X, Y]$ is a local frame for $\mathcal{E}$ and the orientation defined by this frame is independent of the choice of $X, Y$. Thus $\mathcal{E}$ has a preferred orientation.

Let $M$ be an oriented manifold with an oriented Engel structure $\mathcal{D}$ and fix an auxiliary Riemannian metric on $M$. We orient $\mathcal{W}$ such that the contact orientation of local transversals followed by the orientation of $\mathcal{W}$ is the orientation of the manifold. We choose an orthonormal framing $W, X, Y, Z$ of $M$ such that $W$ spans and orients $\mathcal{W}$, $X$ spans and orients the orthogonal complement of $\mathcal{W}$ in $\mathcal{D}$, $Y$ spans the orthogonal complement of $\mathcal{D}$ in $\mathcal{E}$ and $Z$ spans and orients the orthogonal complement of $\mathcal{E}$ in $TM$.

This proves Proposition 1.1. We refer to the framings we have just constructed as *Engel framings.* All Engel framings associated to $\mathcal{D}$ are homotopic since they depend only on the choice of the Riemannian metric.

We finish this section explaining a geometric interpretation from [6] of the condition that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ is a bundle of rank 3. Suppose that the hypersurface $H$ is transverse to $\mathcal{W}$ and let $W$ be a vector field tangent to $\mathcal{W}$ which does not vanish along $H$. Consider the image $H_t$ of $H$ under the local flow $\varphi_t$ of $W$ at time $t$. By the definition of $\mathcal{W}$, the flow $\varphi_t$ preserves $\mathcal{E}$. Then the image of the intersection line field on $H_t$ under $\varphi_{-t}$ is a Legendrian line field on $H$ which rotates without stopping as $t$ increases.

## 3. Outline of the proof of Theorem 1.2

We give only a short outline of the proof of Theorem 1.2. Detailed proofs of the results presented in this note will be published in [9].

For open 4–manifolds with trivial tangent bundle one can use Gromov's $h$–principle for open differential relations which are invariant under the action of the group of diffeomorphisms of $M$, cf. [2]. This yields Engel structures on open 4–manifolds with trivial tangent bundle. From now on we consider a closed manifold $M$. In this situation, the proof relies on round handle decompositions.

A round handle of dimension $n$ and index $k \in \{0, \ldots, n-1\}$ is defined to be $R_k = D^k \times D^{n-k-1} \times S^1$. Round handles are attached to manifolds with boundary using embeddings of $\partial_- R_k = \partial D^k \times D^{n-k-1} \times S^1$ into $\partial M$. We say that $M$ admits a round handle decomposition if $M$ can be obtained from the disjoint union of several round handles of index 0 by successively attaching round handles.

**Theorem 3.1 (Asimov [1])** *A closed connected manifold of dimension $n \neq 3$ admits a round handle decomposition if and only if its Euler characteristic vanishes.*

The analogous statement in dimension 3 is wrong [7].

For the proof of Theorem 1.2 we fix model Engel structures on round handles of dimension 4 whose characteristic foliation $\mathcal{W}$ is transverse to $\partial_- R_k$ and $\partial_+ R_k = \partial R_k \setminus \partial_- R_k$ and $\mathcal{W}$ is oriented such that it points outwards along $\partial_+ R_k$ and inwards along $\partial_- R_k$ for $k = 0, 1, 2, 3$. One of the main difficulties in our proof is to construct sufficiently many model Engel structures on round handles. It is actually possible to choose the model Engel structures such that the contact structure on $\partial_- R_k$ depends only on the index.

*Example 2 On $R_1 = D^1 \times D^2 \times S^1$ we use the coordinates $x$ on $D^1$, $y_1, y_2$ on $D^2$ and $t$ on $S^1$. The distribution spanned by*

$$W = \frac{\partial}{\partial t} + \frac{y_1}{2}\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} - \frac{x}{2}\frac{\partial}{\partial x}$$
$$X = \cos(t)\left(y_2\frac{\partial}{\partial y_1} - \frac{\partial}{\partial x}\right) + \sin(t)\left(\frac{x}{2}\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)$$

*is a model Engel structure on $R_1$. The characteristic foliation is spanned and oriented by $W$.*

Let $M$ be a closed connected manifold with trivial tangent bundle. In particular, the Euler characteristic of $M$ vanishes and we can fix a round handle decomposition by Theorem 3.1. We may assume that there is only one round handle of index 0 respectively 3 and that the round handles are ordered according to their index. We shall construct an Engel structure on $M$ by successively attaching round handles and extending an Engel structure to the new round handlebody.

Suppose that we have constructed an oriented Engel structure on the union $M' \subset M$ of the first $l$ round handles such that the characteristic foliation is transverse to $\partial M'$ and the induced contact structure on $\partial M'$ is overtwisted. Since we assumed that there is only one round 0–handle, $\partial M'$ is connected and we can orient the characteristic foliation such that it points outwards along the boundary. In addition, we assume that the Engel framing extends from $M'$ to $M$. This is a necessary condition for the existence of an Engel structure on $M$ which is an extension of the Engel structure on $M'$.

Now we need to attach the next round handle $R_k$. Let $\psi : \partial_- R_k \to \partial M'$ be an embedding and equip $R_k$ with a model Engel structure. If $\psi$ preserves the contact structures together with their orientations and the homotopy class of the intersection line fields, then it is possible to obtain an Engel structure on $M' \cup_\psi R_k$ such that the boundary is transverse to the characteristic foliation. The Engel structure on $M \cup_\psi R_k$ coincides with the model Engel structure on $R_k$ and its restriction to $M'$ is homotopic through Engel structures to the Engel structure on $M'$ such that the even contact structure is constant.

In order to find an attaching map for $R_k$ which satisfies these conditions, we isotope $\psi$ and we choose the model Engel structure on $R_k$ suitably. For the construction of the isotopy one uses several tools from the theory of contact structures like stabilization of Legendrian curves and convex surfaces. At this point the flexibility of overtwisted contact structures is useful.

Recall that the contact structures on $\partial_- R_k$ induced by all our model Engel structures are equivalent. Once we have achieved that the attaching map of $R_k$ preserves contact structures, we choose the model Engel structure on $R_k$ suitably in order to ensure that the attaching map preserves the orientations of the contact structures and the homotopy class of the intersection line field. The proof that there is such a model Engel structure uses the fact that the Engel framing extends from $M'$ to $M' \cup R_k$.

In order to iterate the construction, it remains to ensure that the contact structure on $M' \cup R_k$ is overtwisted and that the Engel framing extends from $M' \cup R_k$ to $M$. The proof in [8,9] of the last assertion relies on the fact that there is only one round 3–handle and it requires that the Engel framing on the union $M_1$ of all round handles of index 0 and 1 is homotopic to the restriction of an auxiliary framing of $M$. It is possible to choose the model Engel structures on $R_0$ and $R_1$ such that the Engel structure on $M_1$ satisfies this condition.

## 4. Connected sums of Engel manifolds

We finish this note with the discussion of another theorem from [8] which can also be proved using round handles and model Engel structures. A detailed exposition will appear in [9].

Notice that the connected sum of two Engel manifolds $M_1, M_2$ does not admit an Engel structure since the Euler characteristic of $M_1 \# M_2$ is $-2$. This can be corrected using an additional summand

$S^2 \times S^2$. When $M_1$ and $M_2$ are parallelizable, then so is $M_1\#M_2\#(S^2 \times S^2)$. In this situation Theorem 1.2 guarantees the existence of an Engel structure on $M_1\#M_2\#(S^2 \times S^2)$. The advantage of the following Theorem 4.1 is that we obtain an Engel structure which is closely related to the original Engel structures on $M_1$ and $M_2$.

**Theorem 4.1** *Let $(M_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ be Engel manifolds such that the characteristic foliations of both Engel structures have closed transversals $H_1$ respectively $H_2$. Then $M_1\#M_2\#(S^2 \times S^2)$ admits an Engel structure $\mathcal{D}$ which coincides with $\mathcal{D}_1$ and $\mathcal{D}_2$ away from tubular neighborhoods of $H_1$ and $H_2$. The characteristic foliation of $\mathcal{D}$ also admits a closed transversal.*

Notice that one can iterate Theorem 4.1. The condition that the characteristic foliation of $\mathcal{D}_i, i = 1, 2$ admits a closed transversal can be replaced by an assumption on the number of full twists of $\mathcal{D}_i$ around a leaf of $\mathcal{W}_i$ in $\mathcal{E}_i$, cf. the interpretation of the condition $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ in Section 2.

In Theorem 4.1 we make no orientability assumption but let us assume for simplicity that the characteristic foliations are oriented. In order to prove Theorem 4.1 we cut $M_1$ along $H_1$ and $M_2$ along $H_2$. We equip $R_1$ with the model Engel structure from Example 2, and $R_2$ with a similar model Engel structure. Then we attach $R_1$ to those boundary components where the characteristic foliation points outwards and we attach $R_2$ to the other boundary components (using an embedding of $\partial_+ R_2$) such that we obtain an Engel manifold whose boundary is transverse to the characteristic foliation and has two diffeomorphic connected components. After one identifies these two components, one obtains a closed Engel manifold which is diffeomorphic to $M_1\#M_2\#(S^2 \times S^2)$.

If $\mathcal{C}$ is a contact structure on a 3–manifold which is trivial as a bundle, then the Engel structure $\mathcal{D}_C$ on $\mathbb{P}\mathcal{C}$ from Example 1 satisfies the hypothesis in Theorem 4.1. Applying Theorem 4.1 to the prolongation of the standard contact structure on $S^3$ one obtains an Engel structure on

$$M_k = (k+1)(S^3 \times S^1)\#k(S^2 \times S^2)$$

for all $k \geq 1$. One can prove that it is impossible to find an Engel structure on $M_k$ using prolongation or the construction of Geiges, although Theorem 1.2 applies, of course.


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