THE KANAI CONNECTION

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In this talk we are going to describe applications of the so-called Kanai connection to Anosov systems. This connection was first introduced in 1978 by H.Hess, who was working on geometric quantization. It was later applied by Kanai to study Anosov systems.

In Section 1, we begin by considering bi-Lagrangian structures on symplectic manifolds and the associated canonical connection. Sections 2 and 3 show how these structures arise if one considers Anosov systems, with smooth Anosov splitting. We sketch applications to the geodesic flow on negatively curved manifolds and to Anosov symplectomorphisms on tori.

1. Bi-Lagrangian structures and the canonical connection

Let M be a $2n$-dimensional manifold and ω a non-degenerate 2-form. For $X, Y \in \mathfrak{X}(M)$ define a vector field $D(X,Y)$ by

$$i_{D(X,Y)}\omega = L_X i_Y \omega. \tag{1}$$

Note that every 1-form α is of the form $i_Z\omega$ for a unique vector field Z, because ω is non-degenerate. We are going to construct a connection from the operator D. First we have the following lemma.

Lemma 1.

$$\begin{aligned} D(fX,Y) &= fD(X,Y), \ \textit{if } \omega(X,Y)=0 \\ D(X,fY) &= fD(X,Y) + (L_X f)Y, \end{aligned}$$

for every $f \in C^\infty(M)$.

Proof. This follows immediately from the definition of D. □

Suppose $\mathcal{F}$ and $\mathcal{G}$ are two complementary smooth distributions on M which are isotropic with respect to ω. Then $\operatorname{rk}\mathcal{F} = \operatorname{rk}\mathcal{G} = \frac{1}{2}\dim M$ and hence the distributions are Lagrangian. We call $(M,\omega,\mathcal{F},\mathcal{G})$ an *almost bi-Lagrangian structure*. Let X_F and X_G denote the projections of a vector field $X \in \Gamma(TM)$ onto $\mathcal{F}$ and $\mathcal{G}$, respectively.

Definition 2. Let X be a vector field on M, Y a section of $\mathcal{F}$. Then

$$\nabla_X Y := D(X_F,Y)_F + ([X_G,Y])_F \tag{2}$$

defines a connection on $\mathcal{F} \to M$.

Proof. Let X be a vector field on M, Y a section of $\mathcal{F}$. Then we have, using the lemma above

$$\begin{aligned}\nabla_{fX}Y &= \nabla_{fX_F}Y + \nabla_{fX_G}Y \\ &= D(fX_F,Y)_F + ([fX_G,Y])_F \\ &= fD(X_F,Y) + (f[X_G,Y] - (L_Y f)X_G)_F \\ &= f(D(X_F,Y) + ([X_G,Y]))_F \\ &= f\nabla_X Y,\end{aligned}$$

and

$$\begin{aligned}\nabla_X(fY) &= \nabla_{X_F}fY + \nabla_{X_G}fY \\ &= D(X_F,fY)_F + ([X_G,fY]) \\ &= fD(X_F,Y)_F + (L_{X_F}f)Y_F \\ &\quad +(f[X_G,Y] + (L_{X_G}f)Y)_F \\ &= f\nabla_X Y + (L_X f)Y.\end{aligned}$$

This shows that the expression above indeed defines a connection on the bundle $\mathcal{F} \to M$. □

Similarly, we can define a connection on the bundle $\mathcal{G} \to M$ by the formula

(3) $$\nabla_X Y := D(X_G,Y)_G + ([X_F,Y])_G,$$

where Y is a section of $\mathcal{G}$.

We have connections on two complementary subbundles of the tangent bundle TM and hence we can form the direct sum connection given by

(4) $$\nabla_X Y := \nabla_X Y_F + \nabla_X Y_G.$$

In the following we call this connection on an almost bi-Lagrangian manifold the *bi-Lagrangian* or *Kanai* connection. It is not difficult to prove the following facts.

1. By construction, ∇ preserves the subbundles $\mathcal{F},\mathcal{G}$.
2. The 2-form ω is parallel with respect to ∇, $\nabla\omega = 0$.
3. The torsion T of ∇ vanishes identically $\Leftrightarrow$ $(M,\omega,\mathcal{F},\mathcal{G})$ is integrable, i.e. ω is symplectic and $\mathcal{F},\mathcal{G}$ are foliations (this is called a *bi-Lagrangian structure* and this is the situation we will consider in the following).
4. TM is flat along every leaf of $\mathcal{F}$ and $\mathcal{G}$.

We have the following basic example.

Example 3. The standard bi-Lagrangian structure on $\mathbb{R}^{2n}$ with coordinates $(x,y) = (x_1,...,x_n,y_1,...,y_n)$ is given by

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \quad \text{and} \quad \mathcal{F} = \{x = const.\}, \mathcal{G} = \{y = const.\}.$$

The curvature of the bi-Lagrangian connection vanishes identically.

Let $(M^{2n},\omega,\mathcal{F},\mathcal{G})$ be a bi-Lagrangian structure. Then we have the following theorem.

Theorem 4 (Hess)**.** *If the curvature vanishes identically, then the bi-Lagrangian structure is locally isomorphic to the standard structure on* $\mathbb{R}^{2n}$.

Proof. Let $p \in M$ be a point and $U \subset M$ a neighbourhood of p diffeomorphic to $\mathbb{R}^{2n}$. By linear algebra, we can choose a basis $e_1, ..., e_n \in \mathcal{F}_p$, $e_{n+1}, ..., e_{2n} \in \mathcal{G}_p$ of T_pM such that

$$\omega(e_i, e_{n+i}) = 1,\ \omega(e_i, e_j) = 0 \text{ otherwise} \quad (*).$$

Using the canonical connection, we can extend these vectors to vector fields $E_1, ..., E_{2n}$ on U. Fact 1 above shows that $E_1, ..., E_n \in \mathcal{F}$ and $E_{n+1}, ..., E_{2n} \in \mathcal{G}$. Because ∇ is parallel, $(*)$ still holds in every point, and because the torsion vanishes we have $[E_i, E_j] = 0$ for all i, j. Hence we can integrate the vector fields to get local coordinates which give the isomorphism with the standard bi-Lagrangian structure on $\mathbb{R}^{2n}$. □

Remark 5. The canonical connection is not necessarily complete. Therefore, we cannot conclude in general that a flat bi-Lagrangian structure on a simply-connected manifold is globally isomorphic to the standard structure on $\mathbb{R}^{2n}$.

2. Geodesic flow of manifolds with $K < 0$

We need some preparations. Let X be a Riemannian manifold and SX the unit sphere bundle. The Liouville 1-form λ on $TX \cong T^*X$ induces a 1-form $\theta = i^*\lambda$ on SX where i denotes the embedding $i : SX \to TX$. We have the following proposition.

Proposition 6. *θ is a contact form on SX and the geodesic spray is the Reeb vector field for θ.*

Here the *Reeb vector field* of a contact form θ is the unique vector field which satisfies $\theta(X) = 1, i_X d\theta = 0$.

Proof. We only sketch the proof of the proposition. To show that θ is contact one can do a direct computation in local coordinates. To prove that the geodesic spray is the Reeb vector field one can use that the geodesic spray is the Hamiltonian vector field for the function $H = \frac{1}{2}\langle v, v\rangle$ on T^*X and the standard symplectic form $d\lambda$. (This in turn can be used to prove the first part of the assertion.) □

Let M be a closed Riemannian manifold with sectional curvature strictly negative, without loss of generality $K \leq -1$. Then the geodesic flow on SM admits an Anosov splitting, given by the horospheric foliations.

Assumption: The Anosov splitting on SM is smooth.

We consider the universal covering $X = \tilde{M}$ and the unit sphere bundle $V = SX$. Let $\mathbb{R} \times V \to V$ be the geodesic flow.

Lemma 7. *The quotient $P = V/\mathbb{R}$ is a manifold and $V \to P$ is a smooth $\mathbb{R}$-bundle.*

Proof. To show that P admits a smooth structure, we give a different description of it. Let $R = \{r : [0, \infty) \to X \mid r \text{ geodesic}\}$ be the set of rays. We call two elements r_1, r_2 of R asymptotic, $r_1 \sim r_2$, if and only if the distance $\mathrm{dist}(r_1(t), r_2(t))$ is bounded for $t \to \infty$. The quotient $B = R/\sim$ is called the imaginary boundary of X at infinity. (Note that, as $X \cong \mathbb{R}^{n+1}$, B is homeomorphic to S^n, where a topological structure on B is induced by the bijection $V_p \to B, v \mapsto \lim_{t\to\infty} exp_p tv$. One can show that the induced topological structure on B does not depend, up to homeomorphism, on the point $p \in X$ chosen.) We claim that $P = (B \times B \setminus diagonal)/\mathbb{Z}_2$. This follows because for distinct $b^-, b^+ \in B$ there exists, due to the assumption $K \leq -1$, a unique geodesic line l of X such that $l(t) \to b^{\pm}$ as $t \to \pm\infty$. If we divide by the $\mathbb{Z}_2$-action permuting b^-, b^+ we get a unique point of P. This shows that P admits a differentiable structure. We can choose this structure such that the projection $V \to P$ is smooth. □

Let θ be the the canonical contact structure on V and u the geodesic spray. Because u is the Reeb vector field of θ, $i_u d\theta = 0$, and $L_u d\theta = i_u dd\theta + di_u d\theta = 0$, hence $d\theta$ is invariant under the geodesic flow. This implies that $d\theta$ descends to a closed 2-form ω on P. Moreover, $d\theta$ is non-degenerate on $ker\theta$, hence ω is symplectic.

Let $TX = \mathbb{R}u \oplus E^{ss} \oplus E^{uu}$ be the Anosov splitting. Then E^{ss} and E^{uu} are foliations which are invariant under the geodesic flow and hence descend to complementary foliations $\mathcal{F}^{ss}, \mathcal{F}^{uu}$ on P. By assumption, this splitting of TP is smooth.

Lemma 8. *The foliations $\mathcal{F}^{ss}, \mathcal{F}^{uu}$ are Lagrangian with respect to ω.*

Proof. We have the inequalities

$$\begin{aligned} |D\phi_t \xi| &\leq const \cdot e^{-t}|\xi| \quad \text{for } \xi \in E^{ss}, t > 0 \\ |D\phi_- t\xi| &\leq const \cdot e^{-t}|\xi| \quad \text{for } \xi \in E^{uu}, t > 0. \end{aligned}$$

Suppose $\xi_1, \xi_2 \in E^{ss}$. Then, since $d\theta$ is invariant under the geodesic flow,

$$\begin{aligned} |d\theta(\xi_1, \xi_2)| &= |(\phi_t^* d\theta)(\xi_1, \xi_2)| = |d\theta(\phi_{t*}\xi_1, \phi_{t*}\xi_2)| \\ &\leq const \cdot |d\theta| \cdot |\phi_{t*}\xi_1| \cdot |\phi_{t*}\xi_2| \\ &\leq const \cdot |d\theta| \cdot |\xi_1| \cdot |\xi_2| \cdot e^{-2t}, \quad \forall t > 0. \end{aligned}$$

Letting $t \to \infty$ we get $d\theta(\xi_1, \xi_2) = 0$. Similarly, $d\theta|_{E^{uu}} = 0$. This shows that $\mathcal{F}^{ss}, \mathcal{F}^{uu}$ are Lagrangian. □

We conclude that $(P, \omega, \mathcal{F}^{ss}, \mathcal{F}^{uu})$ is a bi-Lagrangian manifold. The splitting of TP is smooth, hence the Kanai connection is defined. This construction can be used to prove the following theorem for the geodesic flow of M (see [5], [3], [2]).

Theorem 9 (Feres, Feres-Katok, Kanai)**.** *Let M be as above. Assume further that*

1. $-4 < K \leq -1$ *or*
2. $dimM$ *is odd.*

Then the geodesic flow of M is C^∞-isomorphic to the geodesic flow of a closed Riemannian manifold M' of constant negative curvature.

Here isomorphic means conjugate under a C^∞-diffeomorphism $SM \to SM'$.

Remark 10. 1. Note that M is homotopy equivalent to M': This is clear for $dimM = 2$. Assume $n = dimM > 2$. The exact homotopy sequence

$$\pi_1(S^{n-1}) \to \pi_1(SM) \to \pi_1(M) \to \pi_0(S^{n-1}),$$

together with the dimension assumption shows that $\pi_1(M) \cong \pi_1(SM)$. Hence, $\pi_1(M) \cong \pi_1(M')$. Because M, M' are aspherical (as their universal covers are $\mathbb{R}^n$) it follows that $M \simeq M'$. A conjecture of Feres-Katok ([3]) says that M is isometric to M'.

2. The first assumption in the theorem is sharp, as one can show by considering Hermitian hyperbolic space $\mathbb{C}H^n$.

Idea of proof of Theorem 9: For M as in the hypotheses of the theorem we consider the associated bi-Lagrangian manifold $(P, \omega, \mathcal{F}^{ss}, \mathcal{F}^{uu})$ with the Kanai connection ∇. We then try to show that, under the conditions on the sectional curvature of the Levi-Civita connection on M, the Kanai connection is affine locally symmetric, i.e. $\nabla R \equiv 0$, where R denotes the curvature of ∇. Then we use the following theorem.

Theorem 11 (Kanai)**.** *If (P, ∇) is locally symmetric, then the geodesic flow of M is isomorphic to the flow of some closed M' with constant negative curvature.*

This theorem is proved by using an algebraic classification of affine locally symmetric spaces with additional structures given by the complementary foliations and the properties of ∇ with respect to these foliations.

The proof that the Kanai connection is locally symmetric is difficult in the general case. We just do it in the following special situation ([5]) (which was the first application of the bi-Lagrangian connection to Anosov systems).

Lemma 12 (Kanai)**.** *If $-\frac{9}{4} < K \leq -1$ then (P, ∇) is locally symmetric.*

Proof. Suppose that $-\lambda^2 \leq K \leq -1$ for some $\lambda \geq 1$. Let ϕ_t denote the geodesic flow of X. We have the following inequalities:

$$\begin{array}{llll} const^{-1} \cdot e^{-\lambda t}|\xi| \leq & |D\phi_t \xi| & \leq const \cdot e^{-t}|\xi| & \text{for } \xi \in E^{ss}, t > 0 \\ const^{-1} \cdot e^{-\lambda t}|\xi| \leq & |D\phi_- t\xi| & \leq const \cdot e^{-t}|\xi| & \text{for } \xi \in E^{uu}, t > 0. \end{array}$$

Using that the splitting is invariant under ϕ_t we get the following inequalities:

$$\begin{array}{llll} const^{-1} \cdot e^{t}|\xi| \leq & |D\phi_- t\xi| & \leq const \cdot e^{\lambda t}|\xi| & \text{for } \xi \in E^{ss}, t > 0 \\ const^{-1} \cdot e^{t}|\xi| \leq & |D\phi_t \xi| & \leq const \cdot e^{\lambda t}|\xi| & \text{for } \xi \in E^{uu}, t > 0. \end{array}$$

Define the tensor $\check{R}$ on the manifold P by $\check{R}(X, Y, Z, T) = \omega(R(X, Y)Z, T)$ and let $S = \pi^*(\nabla \check{R})$, where $\pi : V \to P$ is the projection. Because $\nabla \omega = 0$ it suffices to show that $S \equiv 0$ if $\lambda < \frac{3}{2}$. Note that $|S|$ is bounded on $V = SX$ because $\pi_1(M)$ acts on X and V, preserving all structures and hence by isomorphisms of $(P, \omega, \mathcal{F}^{ss}, \mathcal{F}^{uu})$. Therefore S is invariant under the action of $\pi_1(M)$ and hence comes from a tensor field on the compact manifold $V/\pi_1(M) = SM$.

Suppose for instance $\xi_1, \xi_2, \xi_3 \in E^{ss}$ and $\eta_1, \eta_2 \in E^{uu}$. Since S is invariant under the geodesic flow,

$$\begin{array}{rcl} |S(\xi_1, \xi_2, \xi_3\, \eta_1, \eta_2)| & = & |(\phi_t^* S)(\xi_1, \xi_2, \xi_3\, \eta_1, \eta_2)| \\ & = & |S(\phi_{t*}\xi_1, \phi_{t*}\xi_2, \phi_{t*}\xi_3\, \phi_{t*}\eta_1, \phi_{t*}\eta_2)| \\ & \leq & const \cdot |S| \cdot |\phi_{t*}\xi_1| \cdot |\phi_{t*}\xi_2| \cdot |\phi_{t*}\xi_3| \cdot |\phi_{t*}\eta_1| \cdot |\phi_{t*}\eta_2| \\ & \leq & const \cdot |S| \cdot |\xi_1| \cdot |\xi_2| \cdot |\xi_3| \cdot |\eta_1| \cdot |\eta_2| \cdot e^{(2\lambda - 3)t}, \quad \forall t > 0. \end{array}$$

Letting $t \to \infty$ this implies $S(\xi_1, \xi_2, \xi_3\, \eta_1, \eta_2) = 0$. The other cases can be done in the same way, hence we get $S \equiv 0$ and $\nabla R \equiv 0$. □

3. Anosov symplectomorphisms on low-dimensional tori

Let (M, ω) be a closed symplectic manifold and $\phi : M \to M$ an Anosov diffeomorphism preserving ω, $\phi^* \omega = \omega$. In the first talk by D. Kotschick it was shown that the Anosov property implies that the stable and unstable foliations, $\mathcal{F}^s, \mathcal{F}^u$, are Lagrangian.

Assumption: The Anosov splitting on M is smooth.

Then $(M, \omega, \mathcal{F}^s, \mathcal{F}^u)$ is a bi-Lagrangian manifold and the canonical connection ∇ is defined. Note that ϕ preserves all structures, hence ϕ is affine with respect to ∇, $\phi_*(\nabla_X Y) = \nabla_{\phi_* X}\phi_* Y$.

The structure of Anosov symplectomorphisms is given by the following theorem which is a corollary of a result by Benoist-Labourie (see [1]).

Theorem 13 (Benoist-Labourie). *Let $\phi : M \to M$ be an Anosov symplectomorphism such that the Anosov splitting is smooth. Then ϕ is C^∞-conjugate to a hyperbolic infranil automorphism.*

If G is a nilpotent Lie group, $\Gamma \subset Aff(G)$ a lattice with smooth quotient $M = G/\Gamma$, then an infranil automorphism of M is a map which comes from an automorphism of G, compatible with the lattice Γ. We are going to consider a special case (see [4]) of this theorem.

Theorem 14 (Flaminio-Katok). *Let $M = T^2$ or T^4 and ϕ and Anosov symplectomorphism such that the Anosov splitting is smooth. Then ϕ is C^∞-conjugate to a linear automorphism of M.*

Idea of proof: One first shows that (M, ∇) is complete and locally symmetric. Then one uses algebra to show that ∇ is in fact flat.

Suppose $M = T^2$ (the case T^4 follows similarly). Lift ϕ to an Anosov symplectomorphism $\tilde{\phi}$ on the universal cover $\widetilde{M} = \mathbb{R}^2$. The associated Kanai connection $\widetilde{\nabla}$ is complete and flat, hence by Theorem 4 we can change coordinates such that the bi-Lagrangian structure on $\widetilde{M}$ is the standard one. The diffeomorphism $\tilde{\phi}$ is Anosov, affine and preserves the coordinate foliations, and therefore has the form

$$\tilde{\phi}(x) = Ax + b, \quad \text{with } A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \neq 1.$$

In particular, $\tilde{\phi}$ has a unique fixed point, hence we can choose coordinates such that $b = 0$.

Let $\tau \in \pi_1(T^2)$, acting on $\widetilde{M}$ by deck transformations. Then τ is affine, preserves $\widetilde{\omega}$ and the coordinate foliations, and hence has a similar form as $\tilde{\phi}$. However, if it is not a pure translation, then it must have a fixed point, which is impossible. Therefore the fundamental group acts by translations, and in the induced coordinates on T^2, the diffeomorphism ϕ is linear. $\square$

References

[1] Y. Benoist, F. Labourie, *Sur les diffeomorphisms d'Anosov affines a feuilletages stable et instable differentiables*, Invent.math. 111 (1993) 285-308

[2] R. Feres, *Geodesic flows on manifolds of negative curvature with smooth horospheric foliations*, Ergod.Th. and Dynam.Sys. 11 (1991) 653-686

[3] R. Feres, A. Katok, *Invariant tensor fields of dynamical systems with pinched Lyapunov exponents and rigidity of geodesic flows*, Ergod.Th and Dynam.Sys. 9 (1989) 427-432

[4] L. Flaminio, A. Katok, *Rigidity of symplectic Anosov diffeomorphisms on low dimensional tori*, Ergod.Th. and Dynam.Sys. 11 (1991) 427-440

[5] M. Kanai, *Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations*, Ergod.Th. and Dynam.Sys. 8 (1988) 215-239