Mathematisches Institut
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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 9

1. a) Show that the tangent bundle of S^3 admits a global trivialization by vector fields E_1, E_2, E_3 such that $[E_i, E_{i+1}] = E_{i+2}$, where indices are mod 3.

b) Let $\theta_1, \theta_2, \theta_3$ be the basis of 1-forms dual to the E_i's. Define a connection ∇ on TS^3 by

$$\nabla E_i = \theta_{i+1} \otimes E_{i+2} - \theta_{i+2} \otimes E_{i+1},$$

(indices mod 3). Write down the matrix of connection 1-forms and compute the curvature matrix.

2. Let M be a differentiable manifold and $E \to M$ the trivial rank 1 vector bundle with a fixed trivialization. Two connections ∇, ∇' are called *gauge-equivalent* if there exists a smooth map $g : M \to \mathbb{R}^*$ such that $\nabla' s = g^{-1} \cdot \nabla(g \cdot s)$ for all sections s of E. Let ∇ be a flat connection. Then in the trivialization of E, the connection 1-form of ∇ is a closed differential form $\alpha \in \Omega^1(M)$. Conversely, every closed 1-form α on M determines a flat connection ∇^α. Show that:

$$\nabla^\alpha,\ \nabla^{\alpha'} \text{ are gauge-equivalent} \Longleftrightarrow [\alpha] = [\alpha'] \in H^1_{DR}(M).$$

3. a) Let $E \to \overline{D^n}$ be a vector bundle. Then E is trivial because $\overline{D^n}$ is contractible. Suppose ∇ is a flat connection on E. Show that E admits a trivialization by parallel sections.

b) A connected manifold M is called *simply-connected* if every smooth map $S^1 \to M$ extends to a smooth map $\overline{D^2} \to M$. Equivalently, every map from S^1 to M is homotopic to a constant map. Show that over simply-connected manifolds every vector bundle which admits a flat connection is trivial.

4. Let (M, g) be an n-dimensional orientable Riemannian manifold and $\{(U_i, \phi_i)\}_{i \in I}$ an oriented atlas. In each chart we have the matrix with entries $g_{kl} = g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l})$ and we can consider its determinant $det(g_{kl})$. Define on U_i the n-form

$$\omega_i = \sqrt{det(g_{kl})}\, dx_1 \wedge \ldots \wedge dx_n.$$

Show that the differential forms ω_i fit together and give a globally well-defined volume form. This canonical volume form is denoted by $dvol_g$ (though by Ex. 3 on sheet 5, volume forms on closed manifolds are never exact).

Please hand in your solutions on Thursday, the 8th of January, in the lecture.