Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 7

1. a) Let M be a closed oriented differentiable manifold and $p \in M$ a point. Show that the constant map $f : M \to M$, $x \mapsto p$, is not homotopic to the identity map.

b) Let $\phi_n : S^1 \to S^1$, $z \mapsto z^n$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the standard volume form ω on S^1. Show that $\phi_n^* \omega = n\omega$ and that the maps ϕ_n are pairwise non-homotopic.

2. Let $\overline{D^n} = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$. Show that every smooth map $f : \overline{D^n} \to \overline{D^n}$ has a fixed point (you can use Exercise 1. from the last problem sheet). This is called the *Brouwer fixed point theorem.* Show that this does not hold for a map from the open ball to itself.

3. Let V be an n-dimensional vector space. A 2-form $\omega \in \Lambda^2 V^*$ is called non-degenerate if ker $\omega = 0$. Suppose ω is non-degenerate. Show that:

a) The dimension of V is even, $n = 2k$, there exists a basis $dx_1, ..., dx_{2k}$ of V^* such that

$$\omega = \sum_{i=1}^{k} dx_{2i-1} \wedge dx_{2i}$$

and ω^k is a volume form.

b) For every 1-form $\alpha \in V^*$ there exists a unique vector $x \in V$ such that $i_x \omega = \alpha$.

4. Let M be a closed differentiable manifold. A 2-form $\omega \in \Omega^2(M)$ is called symplectic if it is closed and non-degenerate at each point.

a) Show that if a closed manifold M admits a symplectic structure then M is orientable and $H^2_{DR}(M)$ is non-zero.

b) Let $\{\omega_t\}_{t \in [0,1]}$ be a family of symplectic forms, depending differentiably on t and such that $[\omega_t] = \text{const.} \in H^2_{DR}(M)$. Show that there exists a diffeomorphism $\phi : M \to M$ isotopic to the identity such that $\omega_0 = \phi^* \omega_1$.

Please hand in your solutions on Thursday, the 11th of December, in the lecture.