Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 6

1. a) Let M, N be differentiable manifolds and $f : M \to N$ smooth. Show that

$$d(f^*\alpha) = f^* d\alpha,$$

for all differential forms α on N.

b) Let M be a compact oriented differentiable manifold with boundary $\partial M \neq \emptyset$. We can choose a volume form ω on ∂M. Show that there is no retraction of M onto ∂M, i.e. no smooth map $r : M \to \partial M$ with $r|_{\partial M} = \mathrm{Id}_{\partial M}$.

2. a) Let M be a smooth manifold without boundary and α a differential form on $M \times [0,1]$. Denote by i_s the injection $M \to M \times [0,1], p \mapsto (p, s)$. Consider the vector field $T = \partial/\partial t$ on $M \times [0,1]$, where t is the coordinate along the intervall $[0,1]$. Prove the following formula:

$$i_1^*\alpha - i_0^*\alpha = \int_0^1 (i_s^* L_T \alpha) ds.$$

b) Let $f : M \to N$ be a smooth map. Show that the map

$$f^* : H^k_{DR}(N) \to H^k_{DR}(M), [\omega] \mapsto [f^*\omega] \quad \text{(for every } k \geq 0)$$

is well defined. Prove that this map is the same for homotopic maps f and g, i.e. it depends only on the homotopy class of f.

3. Let M be a (connected) differentiable manifold and $\alpha \in \Omega^1(M)$ a 1-form. For a smooth map $\phi : S^1 \to M$ define

$$\int_\phi \alpha := \int_{S^1} \phi^* \alpha.$$

Show that

$$\alpha \text{ is exact} \Longleftrightarrow \int_\phi \alpha = 0 \quad \text{for all} \quad \phi : S^1 \to M.$$

4. Let M be a connected, oriented differentiable manifold of dimension n. The i-th Betti number of M is defined as $b_i(M) = \dim H^i_{DR}(M)$. One can show that for closed M these dimensions are finite. In particular, $b_0(M) = 1$ and $b_n(M) = 1$. Prove for $M = T^4$ the following lower bounds for the remaining non-trivial Betti numbers:

$$b_1(T^4) \geq 4, \; b_2(T^4) \geq 6, \; b_3(T^4) \geq 4.$$

Please hand in your solutions on Thursday, the 4th of December, in the lecture.