Mathematisches Institut
Universität München

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Office hour: Mon 1-2 pm, Room 308 WS 03/04

Geometry of Manifolds I (Prof. Kotschick)
Problem sheet 5

1. The kernel of a k-form $\alpha \in \Lambda^k V^*$ on a vector space V is defined as

$$\ker \alpha = \{v \in V | i_v \alpha = 0\}.$$

a) Let α be a non-zero 1-form and γ a 2-form on V. Show that

$$\begin{aligned} \exists 1\text{-form } \beta \text{ such that } \gamma = \alpha \wedge \beta \quad &\Leftrightarrow \quad \alpha \wedge \gamma = 0 \\ &\Leftrightarrow \quad \gamma(v,w) = 0 \quad \forall v, w \in \ker \alpha. \end{aligned}$$

b) Let M be a differentiable manifold and let $\alpha \in \Omega^1(M)$ be a 1-form. The kernel $D = \ker \alpha$ of α is defined as the set of all vector fields X on M such that $i_X \alpha = 0$. D is called integrable if $[X,Y] \in D$ for all $X, Y \in D$. Show that

$$D \quad \text{is integrable} \Longleftrightarrow \exists 1\text{-form } \beta : d\alpha = \alpha \wedge \beta.$$

2. Let M be an n-dimensional differentiable manifold. Show that the following conditions are equivalent:

$a.$) M is orientable

$b.$) $\Lambda^n T^* M$ is a trivial vector bundle

$c.$) There exists a nowhere vanishing n-form $\omega \in \Omega^n(M)$.

A differential form as in $c.$) is called a *volume form*.

3. Let M be a differentiable, closed (i.e. compact, without boundary) orientable manifold of dimension n. Show that $H^n_{DR}(M) \neq 0$.

4. Let M^m, N^n be a differentiable manifolds, where M is closed and orientable. Smooth maps $f, g : M \to N$ are called homotopic if there exists a smooth map $H : M \times [0,1] \to N$ such that $H(-,0) = f,\ H(-,1) = g$. Let $\omega \in \Omega^m(N)$ be a closed m-form on N. Show that

$$\int_M f^*\omega = \int_M g^*\omega.$$

Please hand in your solutions on Thursday, the 27th of November, in the lecture.